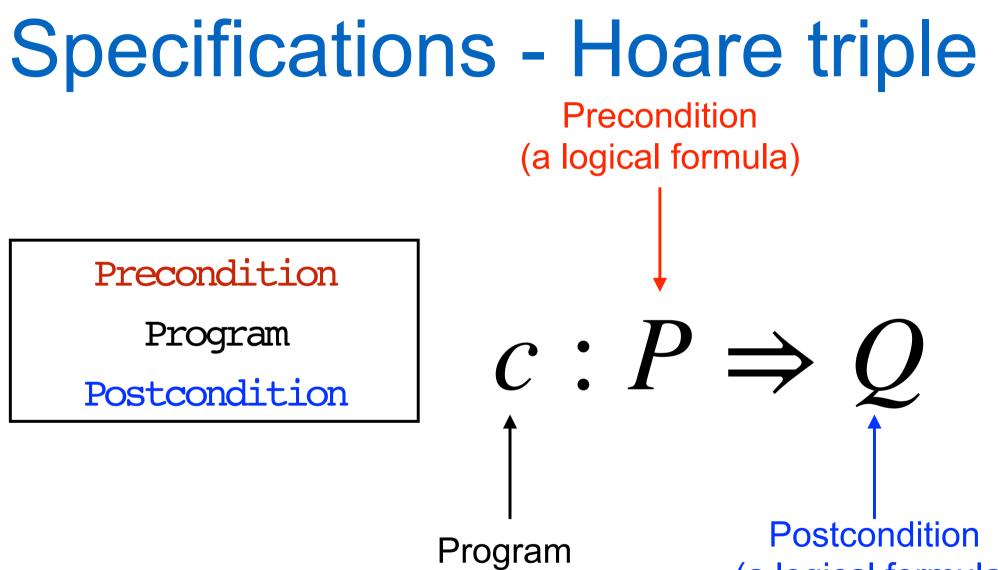
CS 599: Formal Methods in Security and Privacy Hoare Triples and Hoare Logic

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(a logical formula)

$\vdash C; C': P \Rightarrow Q$

⊢c:P⇒R

$$\vdash C; C': P \Rightarrow Q$$

 $\vdash_{\mathbf{C}} : \mathbf{P} \Rightarrow \mathbf{R} \qquad \vdash_{\mathbf{C}} : \mathbf{R} \Rightarrow \mathbf{Q}$

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$$\vdash C; C': P \Rightarrow Q$$

Is this correct?

$Final equation Some Instances \\ Final equation is for a constraint of the second state of the second st$

Is this a valid triple?

Some Instances
$$\vdash x := z * 2; z := x * 2$$
 $: \{(z * 2) * 2 = 8\} \Rightarrow \{z = 8\}$ Is this a valid triple?

Some Instances

How can we prove it?

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For every m such that P(m) and m' such that $\{c, c'\}_m = m'$ we need Q(m').

By our semantics: { c; c' } m=m' if and only if there is m'' such that { c } m=m'' and { c' } m''=m'.

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Assuming $c: P \Rightarrow R$ and $c': R \Rightarrow Q$ valid, if P (m) we can show R (m'') and if R (m'') we can show Q(m'), hence since we have P (m) we can conclude Q(m').

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Can we prove it with the rules that we have?

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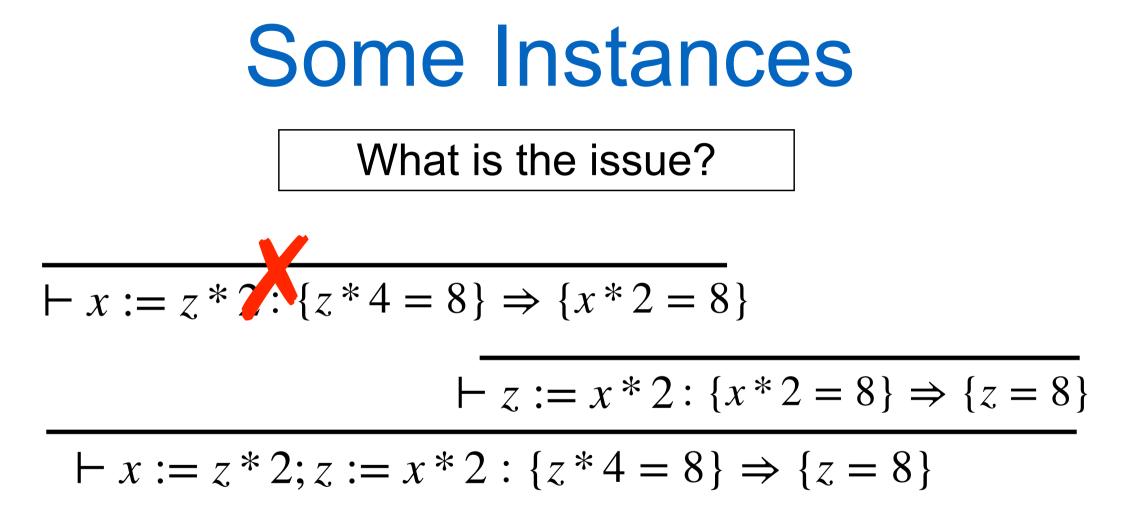
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Rules of Hoare Logic Consequence

$$P \Rightarrow S \qquad \vdash c : S \Rightarrow R \qquad R \Rightarrow Q$$

$$\vdash C: P \Rightarrow Q$$

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Rules of Hoare Logic If then else

$\vdash if e then c_1 else c_2 : P \Rightarrow Q$

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Is this correct?

⊢ if y = 0 then skip else x := x + 1; x := x - 1: {x = 1} ⇒ {x = 1}

Is this a valid triple?

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Is this a valid triple?

Can we prove it with the rules that we have?

Some Instances

 $\vdash \texttt{skip:} \{x = 1\} \Rightarrow \{x = 1\} \quad \vdash x := x + 1; x := x - 1 : \{x = 1\} \Rightarrow \{x = 1\}$

•

⊢ if y = 0 then skip else x := x + 1; x := x - 1: {x = 1} ⇒ {x = 1}

Rules of Hoare Logic If then else

⊢cı:P⇒Q

 $\vdash c_2 : P \Rightarrow Q$

 $\vdash if e then c_1 else c_2 : P \Rightarrow Q$

Rules of Hoare Logic If then else



 $\vdash if e then c_1 else c_2 : P \Rightarrow Q$

Is this strong enough?

⊢ if false then skip else x = x + 1: {x = 0} ⇒ {x = 1}

Is this a valid triple?

⊢ if false then skip else x = x + 1: {x = 0} ⇒ {x = 1}

Is this a valid triple?



⊢ if false then skip else x = x + 1: {x = 0} ⇒ {x = 1}

Is this a valid triple?

Can we prove it with the rules that we have?

⊢ if false then skip else x = x + 1: {x = 0} ⇒ {x = 1}

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Rules of Hoare Logic If then else

$$\vdash c_1: e \land P \Rightarrow Q \qquad \vdash c_2: \neg e \land P \Rightarrow Q$$

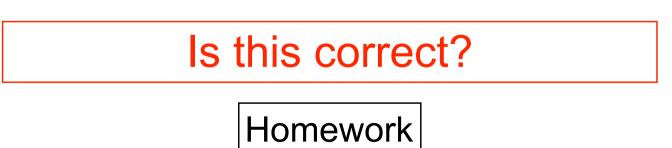
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Rules of Hoare Logic If then else

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\vdash Abort: $? \Rightarrow ?$

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What can be a good specification?

Validity of Hoare triple We say that the triple c: P⇒Q is valid if and only if for every memory m such that P(m) and memory m' such that {c}_m=m' we have Q(m').

⊢Abort:P⇒Q

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Vacuously True

 \vdash while e do c : ??

$$P \Rightarrow \neg e$$

 $\vdash while e do c : P \Rightarrow P$

 $P \rightarrow e \qquad \vdash c : P \rightarrow P$

 $\vdash while e do c : P \Rightarrow P$

 \vdash c : e \land P \Rightarrow P

⊢while e do c : P ⇒ P ∧ ¬e Invariant

$\vdash \text{ while } x = 0 \text{ do } x := x + 1$ $: \{x = 1\} \Rightarrow \{x = 1\}$

How can we derive this?

$\vdash \text{ while } x = 0 \text{ do } x := x + 1$ $: \{x = 1\} \Rightarrow \{x = 1\}$

What can be a good Invariant?

$\vdash \text{ while } x = 0 \text{ do } x := x + 1$ $: \{x = 1\} \Rightarrow \{x = 1\}$

What can be a good Invariant?

 $Inv = \{x = 1\}$

 $\vdash \text{ while } x = 0 \text{ do } x := x + 1 \text{: } \{x = 1\} \Rightarrow \{x = 1 \land x \neq 0\} \qquad x = 1 \land x \neq 0 \Rightarrow x = 1$

 $\begin{array}{ll} x = 1 \land x = 0 \Rightarrow x + 1 = 1 & \vdash x := x + 1 : \{x + 1 = 1\} \Rightarrow \{x = 1\} \\ & \vdash x := x + 1 : \{x = 1 \land x = 0\} \Rightarrow \{x = 1\} \\ & \vdash \text{ while } x = 0 \text{ do } x := x + 1 : \{x = 1\} \Rightarrow \{x = 1 \land x \neq 0\} & x = 1 \land x \neq 0 \Rightarrow x = 1 \end{array}$

 $\begin{array}{l} x = 1 \land x = 0 \Rightarrow x + 1 = 1 \\ \vdash x := x + 1 : \{x = 1\} \Rightarrow \{x = 1\} \\ \vdash x := x + 1 : \{x = 1\} \land x = 0\} \Rightarrow \{x = 1\} \\ \vdash \text{ while } x = 0 \text{ do } x := x + 1 : \{x = 1\} \Rightarrow \{x = 1\} \land x \neq 0\} \\ \begin{array}{l} x = 1 \land x \neq 0 \Rightarrow x = 1 \\ x = 1 \land x \neq 0 \Rightarrow x = 1 \\ \end{array}$

Another example

 $\begin{array}{c|c} x := 3; \\ y := 1; \\ \text{while } x > 1 \ \text{do} \\ y := y + 1; \\ x := x - 1; \end{array} \hspace{0.5cm} : \{true\} \Rightarrow \{y = 3\} \\ \end{array}$

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Another example

 $\begin{array}{c|c} x := 3; \\ y := 1; \\ \text{while } x > 1 \ \text{do} \\ y := y + 1; \\ x := x - 1; \end{array} \begin{array}{c} : \{true\} \Rightarrow \{y = 3\} \end{array}$

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Another example

 $\begin{array}{c|c} x := 3; \\ y := 1; \\ \text{while } x > 1 \ \text{do} \\ y := y + 1; \\ x := x - 1; \end{array} \begin{array}{c} : \{true\} \Rightarrow \{y = 3\} \end{array}$

What can be a good Invariant?

$$Inv = \{y = 4 - x \land x \ge 1\}$$

How do we know that these are the right rules?

Soundness

If we can derive $\vdash_{C} : P \Rightarrow Q$ through the rules of the logic, then the triple $C : P \Rightarrow Q$ is valid.

Are the rules we presented sound?

Completeness

If a triple $C : P \Rightarrow Q$ is valid, then we can derive $\vdash C : P \Rightarrow Q$ through the rules of the logic.

Are the rules we presented complete?

Relative Completeness $P \Rightarrow S$ $\vdash c: S \Rightarrow R$ $R \Rightarrow Q$

$$\vdash C: P \Rightarrow Q$$

Relative Completeness $P \Rightarrow S$ $\vdash c: S \Rightarrow R$ $R \Rightarrow Q$

If a triple $c : Pre \Rightarrow Post$ is valid, and we have an oracle to derive all the true statements of the form $P\RightarrowS$ and of the form $R\RightarrowQ$, which we can use in applications of the conseq rule, then we can derive $\vdash c : Pre \Rightarrow Post$ through the rules of the logic.