Monotonicity testing and isoperimetric inequalities

"On monotonicity testing and Boolean isoperimetric type theorems"

[Khot, Minzer, Safra '15]

"Improved testing algorithms for monotonicity"

[Dodis, Goldreich, Lehman, Raskhodnikova, Ron, Samorodnitsky '99]

"Testing monotonicity"

[Goldreich, Goldwasser, Lehman, Ron, Samorodnitsky '00]

Iden Kalemaj

Boston University

The problem

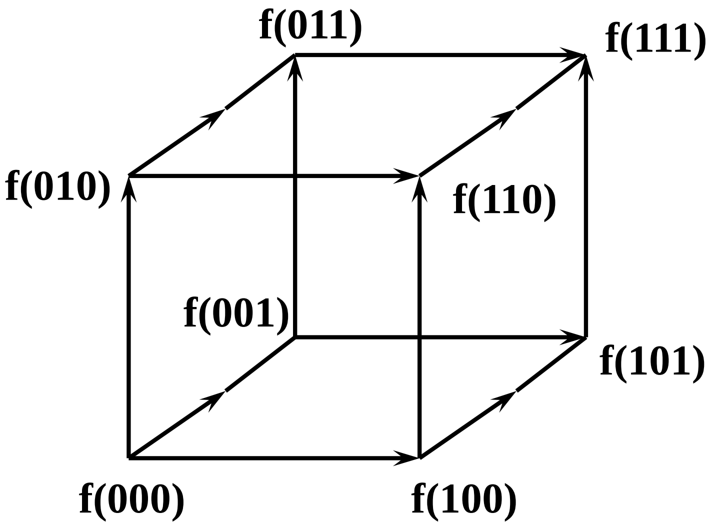
- Query a function $f: \{0,1\}^n \rightarrow \{0,1\}$ at a few points and decide if the function is monotone or far from monotone.
- First introduced by: Goldreich, Goldwasser, Lehman, Ron '98

- "few" = sublinear in the size of the domain
- property testing, sublinear algorithms

Some definitions & Background

Monotonicity on hypercube

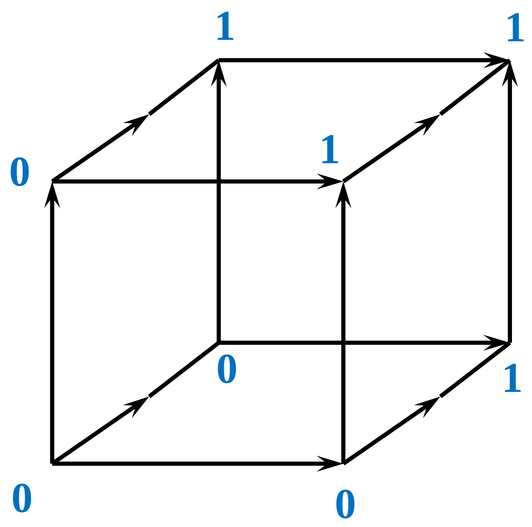
- $f: \{0,1\}^n \rightarrow \{0,1\}$
- $x \rightarrow y$ is an edge if:
  - $x_i = 0, y_i = 1$
  - $x_j = y_j$ for all $j \in [n] - i$



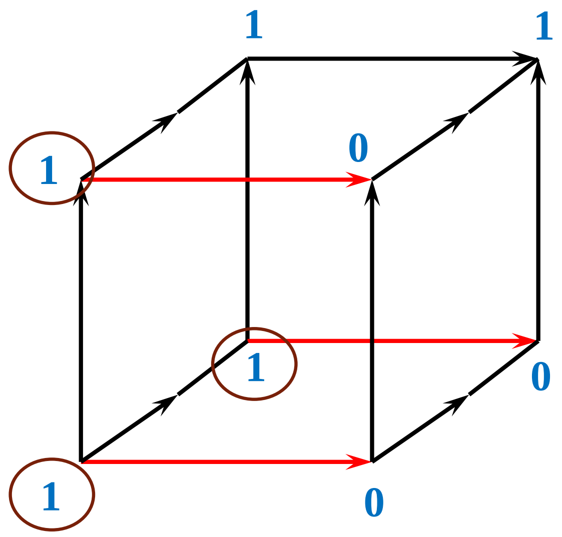
- 2^n vertices and $n \cdot 2^{n-1}$ edges in the hypercube
- f is monotone if the value of f along any edge is nondecreasing

Distance to monotonicity

- Let $\boldsymbol{\varepsilon(f)}$ denote the distance of f to monotonicity
- $\varepsilon(f)$ = least fraction of values of f that need to be changed to make f monotone

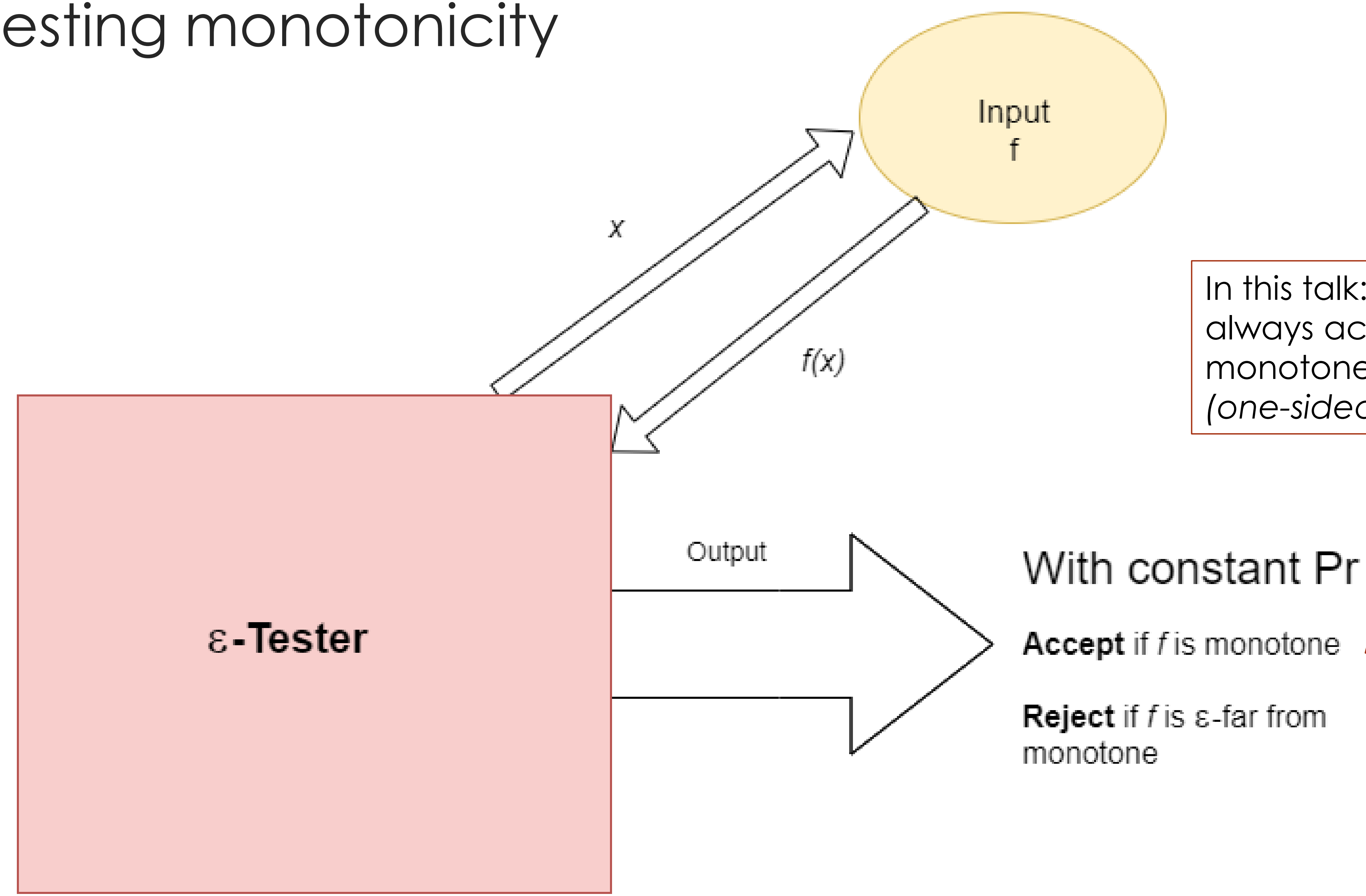


$\mathrm{dist}(f, \mathrm{MONO}) = 0$



$\mathrm{dist}(g, \mathrm{MONO}) = 3/8$

Testing monotonicity



In this talk: tester always accepts if f monotone
(one-sided error)

Results on monotonicity testing

- Variations of problem studied since the late '90s:
  - on different ranges, different domains
  - estimating distance to monotonicity
- For Boolean functions on hypercube:
  - $O\left(\frac{n}{\varepsilon}\right)$-query tester [Dodis, Goldreich, Lehman, Raskhodnikova, Ron, Samorodnitsky '99], [Goldreich, Goldwasser, Lehman, Ron, Samorodnitsky '00]
  - $O\left(\frac{n^{7/8}}{\varepsilon^{3/2}}\right)$-query tester [Chakrabarty, Seshadri '13]
  - $\tilde{O}\left(\frac{\sqrt{n}}{\varepsilon^2}\right)$-query tester [Khot, Minzer, Safra '15]
- Lower bounds:
  - $\Omega(\sqrt{n})$ queries for 1-sided, nonadaptive [Fischer, Lehman, Newman, Raskhodnikova, Rubinfeld, Samorodnitsky '02]:
  - $\widetilde{\Omega}\left(n^{1/3}\right)$ queries for adaptive [Chen, Waingarten, Xie '17]

Plan for this talk

- $O\left(\frac{n}{\varepsilon}\right)$ query tester + analysis
- Overview of isoperimetric inequalities related to monotonicity testing
- Proof outline for isoperimetric inequality in $\tilde{O}\left(\frac{\sqrt{n}}{\varepsilon^2}\right)$-query tester
- Relationship between isoperimetric inequality and $\tilde{O}\left(\frac{\sqrt{n}}{\varepsilon^2}\right)$-query tester

Part 1: Edge tester

- Describe $O\left(\frac{n}{\varepsilon}\right)$ query tester (a.k.a edge tester)
- Analysis of tester

The edge tester [Dodis, Goldreich, Lehman, Raskhodnikova, Ron, Samorodnitsy '99]

Given n and ε:

Repeat $O\left(\frac{n}{\varepsilon}\right)$ times:

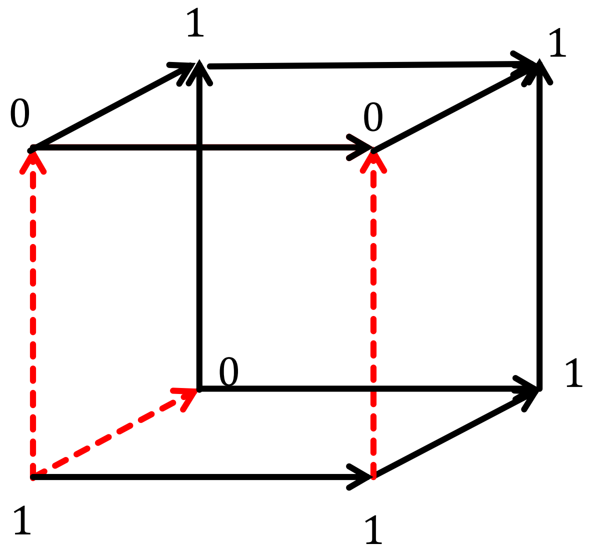
- Sample edge $x \to y$ from the hypercube
- Query $f(x)$ and $f(y)$
- Reject if and only if $f(x) > f(y)$

Accept

- The tester is nonadaptive
- The tester always accepts when f is monotone
- Need to show that tester rejects w.h.p if f is ε-far from monotone

The edge tester: analysis overview

- Want to show that tester rejects w.h.p
- Call edge $x \to y$ is **violated** if:
  - $f(x) = 1,\ f(y) = 0$, i.e. f decreases along the edge

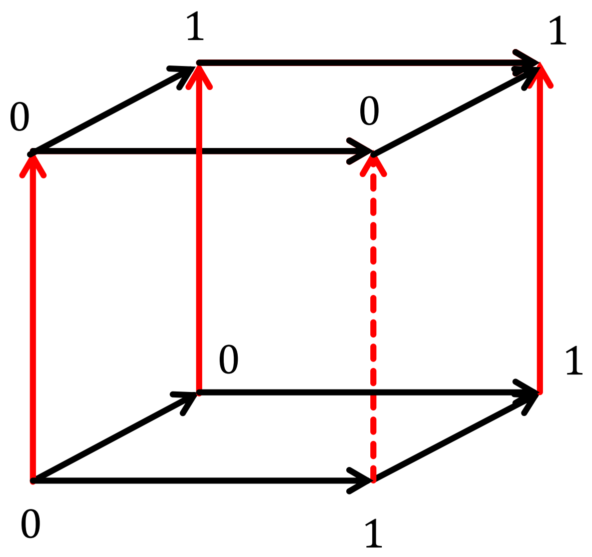


- We show there must be a lot of violated edges:

$$\frac{\#\ \ violated\ edges}{n \cdot 2^{n-1}} \geq \frac{\varepsilon(f)}{n}$$

- If f is ε-far from monotone, tester finds a violated edge w.h.p
- Idea: Can repair f by changing 2 values per violated edge

Switch operator



- $f \rightarrow S_i(f)$
- For all edges along dimension i:

  if the edge $x \rightarrow y$ is violated:

  switch around the values of $f(x)$ and $f(y)$

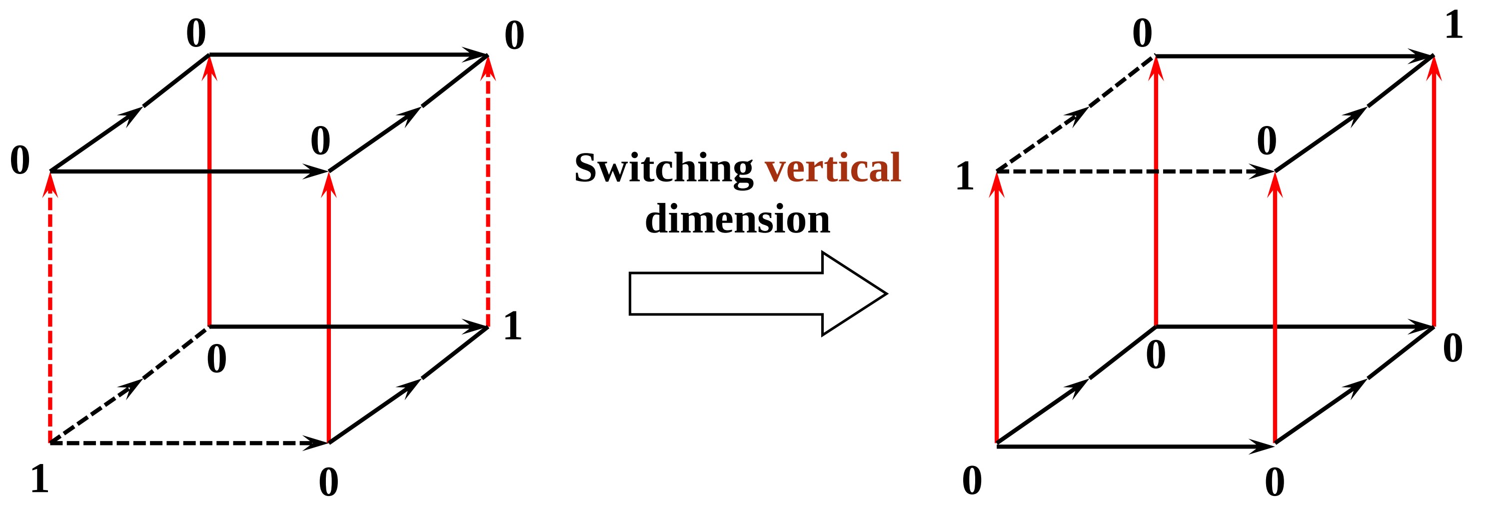
- More precisely:

$i = 1$ $f(y) = b$ $S_i f(y) = \max(a, b)$

$i = 0$ $f(x) = a$ $S_i\, f(x) = \min(a, b)$

Switch operator: example

- Switch the red edges
- Edges in - - - - are violated



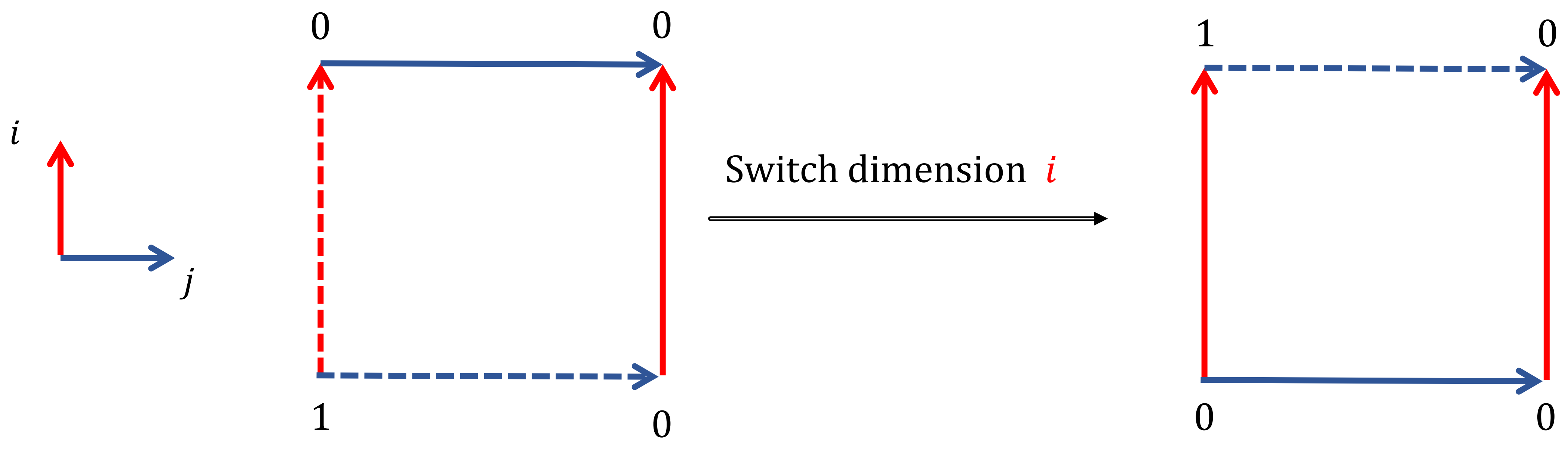
Property of switch operator

Lemma.

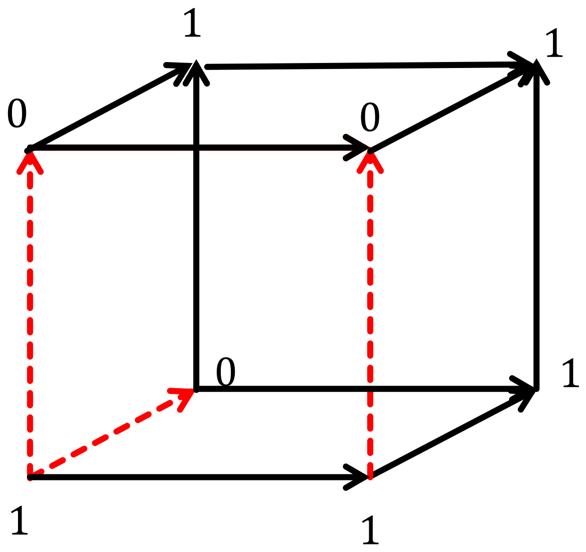
Switching f in dimension i:

- makes f monotone in dimension i
- does not increase number of violated edges in dimension j

Proof. It suffices to look at squares in dimensions i and j



Edge tester: analysis



- $S_1 S_2 \dots S_n(f)$ is monotone
- When switching f in dimensions 1 through n we change at most:

$2 \cdot (\# \text{ violated edges})$ points

- Therefore:

$$\frac{2 \cdot (\# \text{ violated edges})}{2^n} \geq \text{dist}(f, S_1 S_2 \dots S_n(f)) \geq \varepsilon(f)$$

- For a random edge $x \rightarrow y$:

$$\text{Prob}[x \rightarrow y \text{ is violated}] \geq \frac{(\# \text{ violated edges})}{n \cdot 2^{n-1}} \geq \frac{\varepsilon(f)}{n}$$

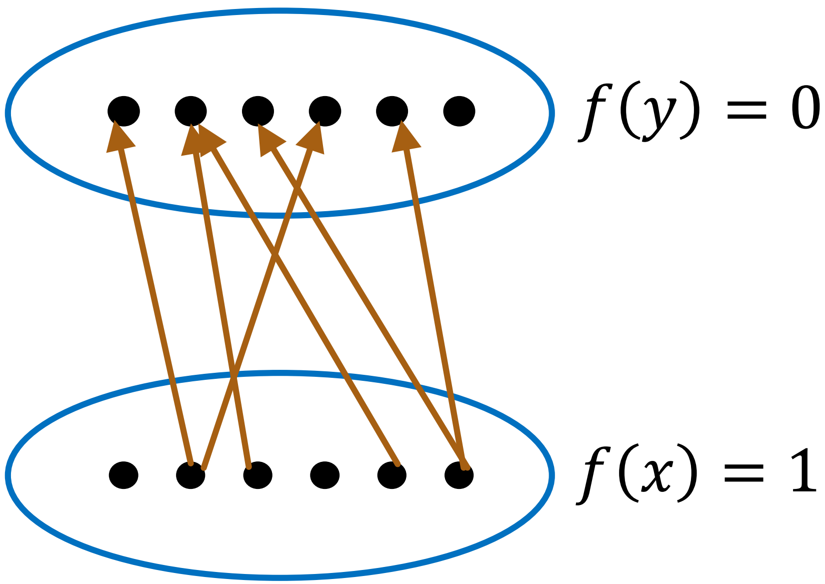
- After $\frac{n}{\varepsilon(f)}$ rounds, w.h.p, we have drawn a violated edge $\rightarrow f$ is rejected

Part 2: Background on isoperimetric inequalities

- Describe isoperimetric inequality of this talk
- Some background on isoperimetric inequalities

Isoperimetric inequality in this talk

Bipartite graph of violated edges



"average square root degree"

$$\frac{\textbf{Sum of square root of degrees of x}}{2^n}$$

$$\geq$$

Distance of f to monotonicity

Isoperimetric inequalities (undirected)

- An edge $x \to y$ is **nonconstant** if $f(x) \neq f(y)$

- Define

$$I_f(x) = \begin{cases} 0 & \text{if } f(x) = 0 \\ \#\ \text{nonconstant edges incident at } x & \text{if } f(x) = 1 \end{cases}$$

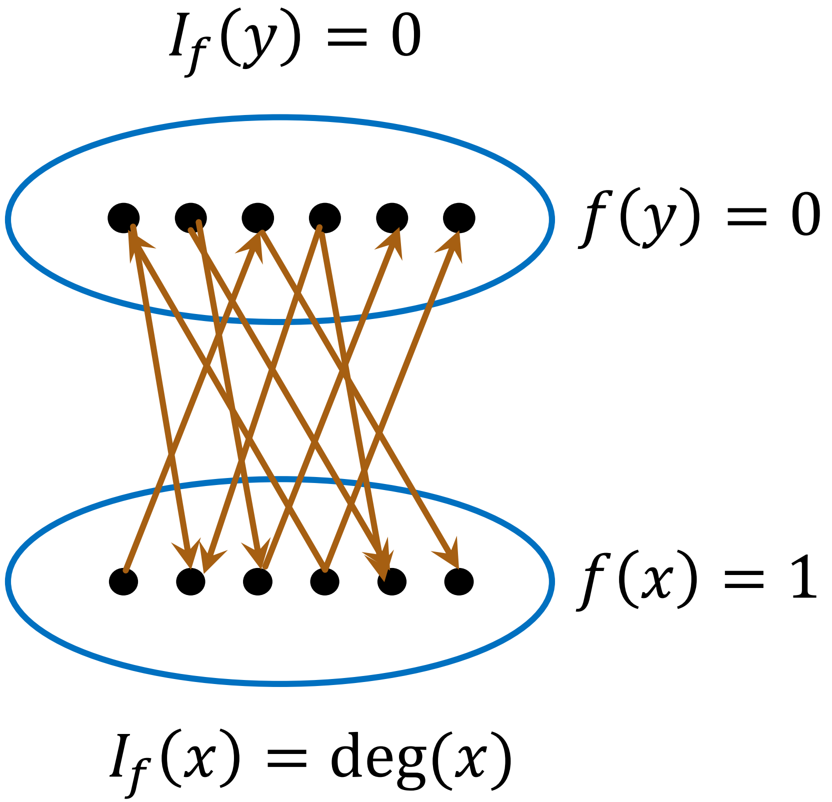
- Then:

$$\mathbf{E}_x\left[I_f(x)\right] \geq \Omega\big(\mathrm{var}(f)\big) \quad \text{[folklore]}$$

$$\mathbf{E}_x\left[\sqrt{I_f(x)}\right] \geq \Omega\big(\mathrm{var}(f)\big) \quad \text{[Talagrand '93]}$$

$\mathrm{var}(f)$ = fraction of ones · fraction of zeroes

Bipartite graph of **nonconstant** edges



Isoperimetric inequalities (directed)

- An edge $x \rightarrow y$ is **violated** if $f(x) > f(y)$
- Define

$$I_f^-(x) = \begin{cases} 0 & \text{if } f(x) = 0 \\ \#\ \text{violated edges incident at } x & \text{if } f(x) = 1 \end{cases}$$

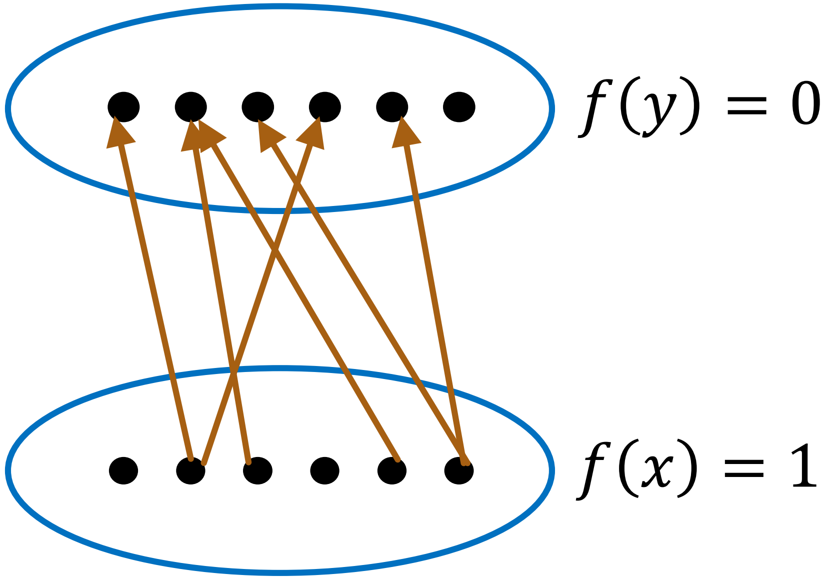
- Then:

$$\mathbf{E}_x\left[I_f^-(x)\right] \geq \Omega\big(\varepsilon(f)\big)$$ [Edge tester]

$$\boxed{\mathbf{E}_x\left[\sqrt{I_f^-(x)}\right] \geq \widetilde{\Omega}\big(\varepsilon(f)\big)}$$ [Khot, Minzer, Safra '15]

Bipartite graph of **violated** edges

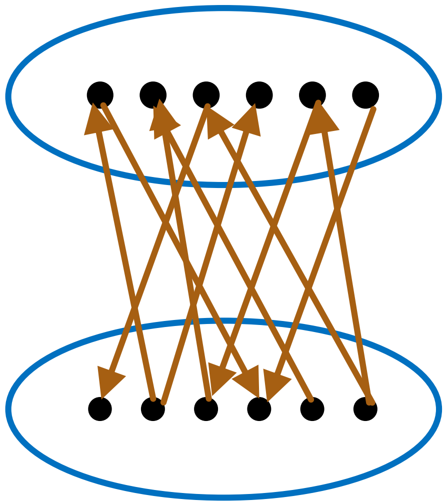
$I_f^-(y) = 0$



$I_f^-(x) = \deg(x)$

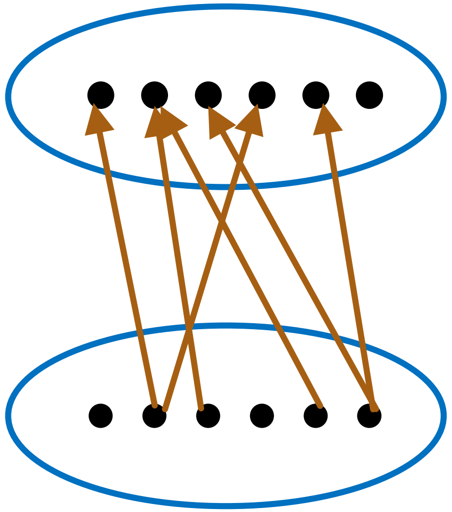
Summary of isoperimetric inequalities

$\mathbf{E}_x[I_f(x)] \geq \Omega(\mathrm{var}(f))$ ←	$\mathbf{E}_x[I_f^-(x)] \geq \Omega(\varepsilon(f))$
$\mathbf{E}_x\left[\sqrt{I_f(x)}\right] \geq \Omega(\mathrm{var}(f))$ ←	$\mathbf{E}_x\left[\sqrt{I_f^-(x)}\right] \geq \tilde{\Omega}(\varepsilon(f))$



$I_f(y) = 0$

$I_f(x) = \deg(x)$



$I_f^-(y) = 0$

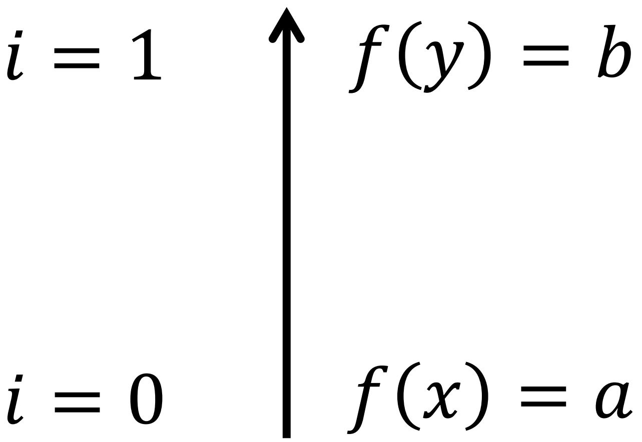
$I_f^-(x) = \deg(x)$

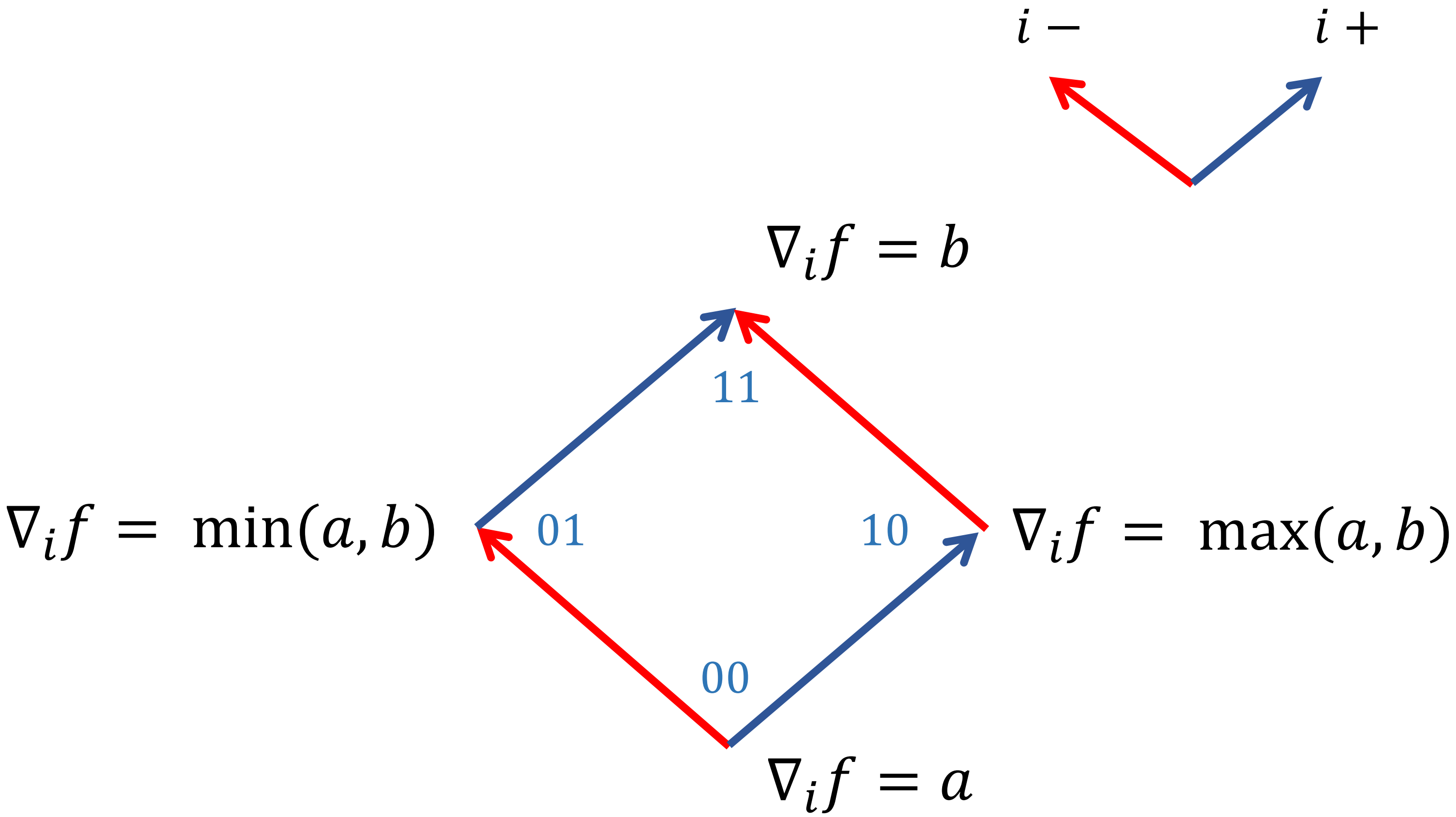
Part 3: Outline of proof of our isoperimetric inequality

- Outline of proof for isoperimetric inequality
  - Only main ideas, no actual proofs!
- But before: define a new operator similar to switch operator
- Relate isoperimetric inequality to analysis of $\sqrt{n}$ - tester

Split operator

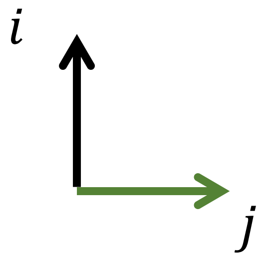
- $f \rightarrow \nabla_i(f)$

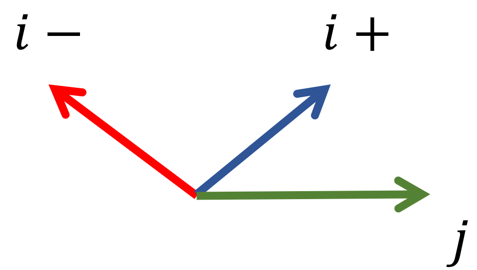


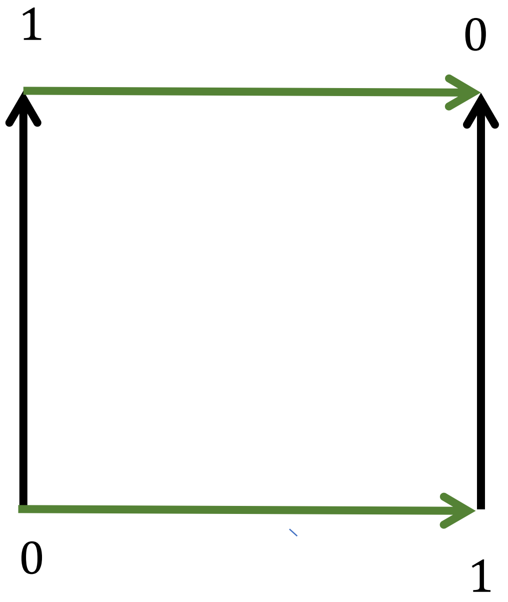


- Non-decreasing (monotone) in dimension $i +$
- Non-increasing in dimension $i -$
- All violated edges will be along dimension $i -$

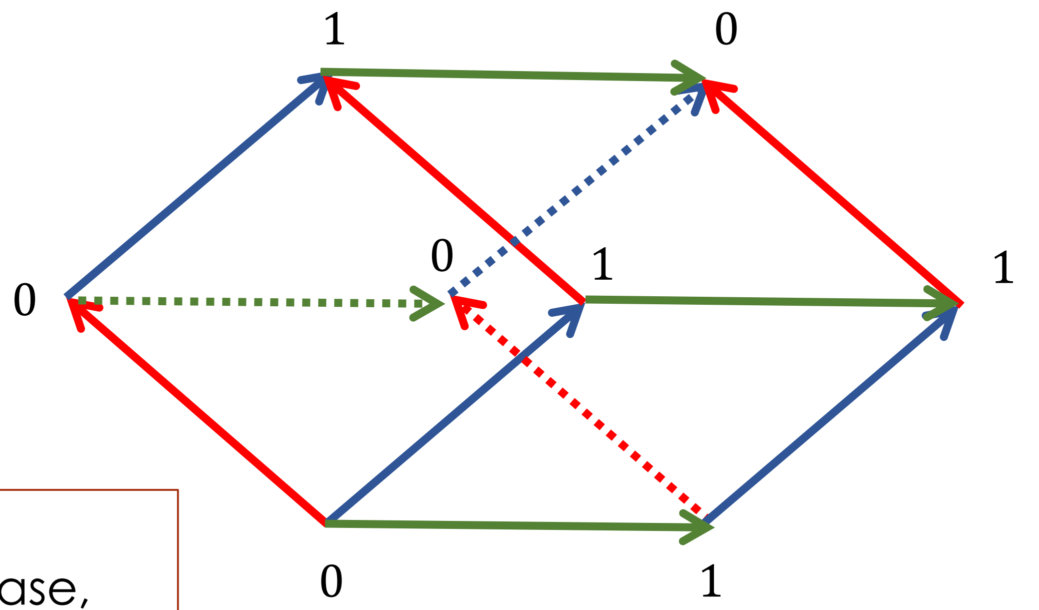
Split operator: example







$\xrightarrow{\text{Split on dimension } i}$



Note:
violated edges may increase,
But they will all be in the negative direction

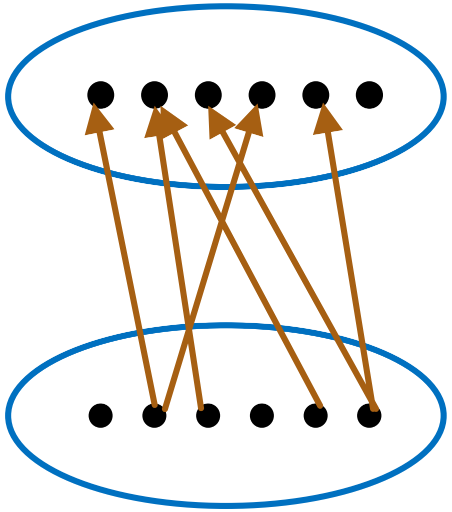
Isoperimetric inequality

$$\boxed{\mathbf{E}_x\left[\sqrt{I_f^-(x)}\right] \geq \widetilde{\Omega}\big(\varepsilon(f)\big)}$$

"average square root degree"

distance to monotonicity

Bipartite graph of violated edges



$f(y) = 0$

$I_f^-(y) = 0$

$f(x) = 1$

$I_f^-(x) = \deg(x)$

Outline of proof: Attempt 1

Objective: $\mathbf{E}_x\left[\sqrt{I_f^-(x)}\right]$

$\geq$

Phase 1:
totally split f to get g: $g = \nabla_1\nabla_2 \ldots \nabla_n(f)$
Splitting only decreases the objective

$\mathbf{E}_x\left[\sqrt{I_g^-(x)}\right]$

$\geq$

Phase 2:
Inequality holds for a **totally split** function

$\varepsilon(g)$

$\geq$

Not true!!

$\varepsilon(f)$

Outline of proof: Attempt 2

Objective: $\mathbf{E}_x\left[\sqrt{I_f^-(x)}\right]$

$\geq$

$\mathbf{E}_x\left[\sqrt{I_g^-(x)}\right]$

$\geq$

$\varepsilon(g)$

$\geq$

$\mathbf{E}[\text{dist}(f, f \text{ switched in } \textbf{all} \text{ the coordinates})]$
$- \mathbf{E}[\text{dist}(f, f \text{ switched in } \textbf{half} \text{ the coordinates})]$

Phase 1:
totally split f to get g: $g = \nabla_1\nabla_2 \ldots \nabla_n(f)$
Splitting only decreases the objective

Phase 2:
Inequality holds for a **totally split** function

Phase 3:
Splitting is like switching half the coordinates

Outline of proof: Attempt 2, <u>Generalized</u>

$$\boxed{\mathbf{E}_x\left[\sqrt{I_f^-(x)}\right]}$$

$$\geq$$

$$\boxed{\mathbf{E}_x\left[\sqrt{I_g^-(x)}\right]}$$

$$\geq$$

$$\boxed{\varepsilon(g)}$$

$$\geq$$

$$\boxed{\begin{array}{c}\mathbf{E}[\mathrm{dist}(f, f \text{ switched in } \mathbf{1/2^i} \text{ of the coordinates})] \\ - \mathbf{E}[\mathrm{dist}(f, f \text{ switched in } \mathbf{1/2^{i+1}} \text{ of the coordinates})]\end{array}}$$

$\log n$ such inequalities!

***g* =**

- $\boldsymbol{f}$ split **only in a subset** of coordinates
- restricted to coordinates that are split
- fixed on rest of coordinates

→ $\boldsymbol{g}$ is **totally split!**

Need to add expectation over order of splits
AND
expectation over values of fixed coordinates

Phase 1 Idea

- **Phase 1:** Splitting only decreases our objective

$$\mathbf{E}_x\left[\sqrt{I_f^-(x)}\right] \geq \mathbf{E}_x\left[\sqrt{I_{\nabla_i f}^-(x)}\right]$$

  - Let $g = \nabla_1\nabla_2 \dots \nabla_n(f)$. Then:

$$\mathbf{E}_x\left[\sqrt{I_f^-(x)}\right] \geq \mathbf{E}_x\left[\sqrt{I_g^-(x)}\right]$$

- **Proof idea:**
  - Like case analysis for switch operator
  - Consider cube in dimensions $i+, i-, j$ instead of a square

Phase 2 Idea

- **Phase 2:** The inequality is true for a "totally split" function

$$\mathbf{E}_x\left[\sqrt{I_g^-(x)}\right] \geq \varepsilon(g)$$

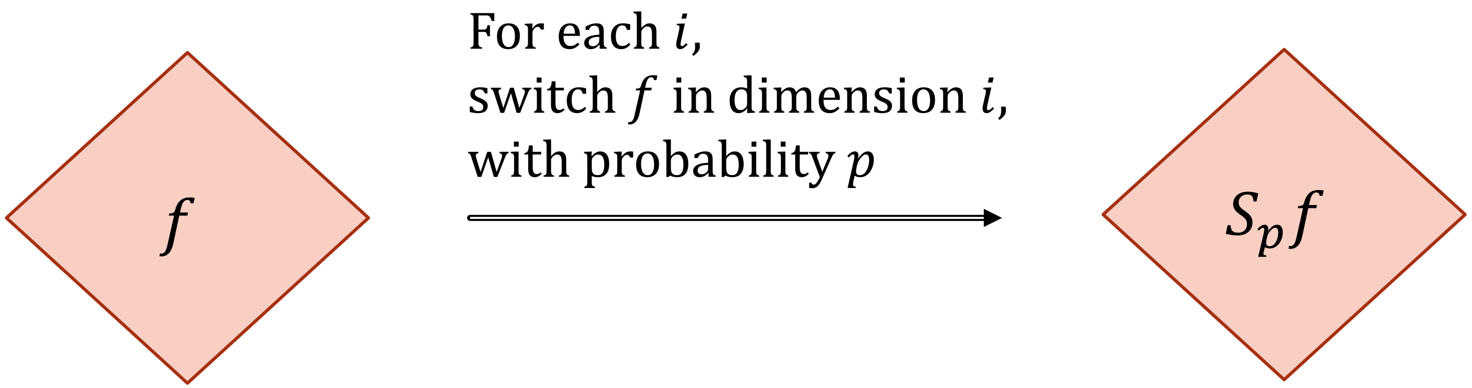
where $g = \nabla_1\nabla_2 \dots \nabla_n(f)$.

- g is "simple": all the violated edges are in the negative coordinates, monotone in half the coordinates
- Use the "undirected" version of the isoperimetric inequality, i.e.:

$$\mathbf{E}_x\left[\sqrt{I_g(x)}\right] \geq \Omega(\mathrm{var}(g))$$

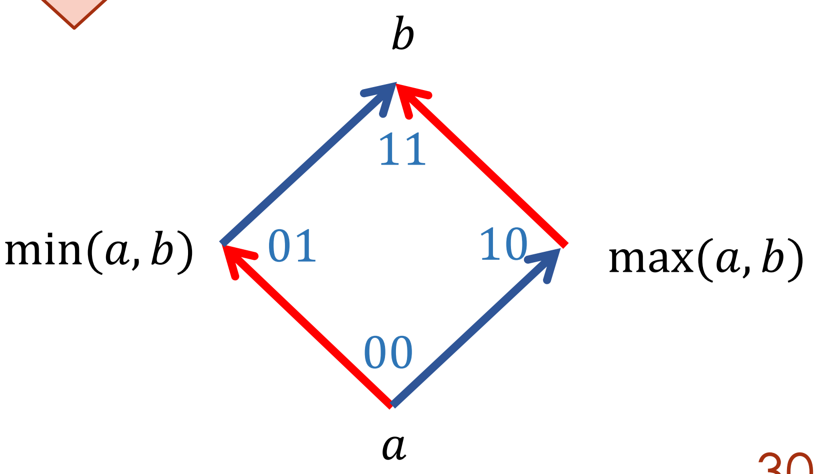
Phase 3 Idea, part 1

- Fix the following order of coordinates: $1, 2, \ldots, n$
- This is the order in which we split f to obtain g

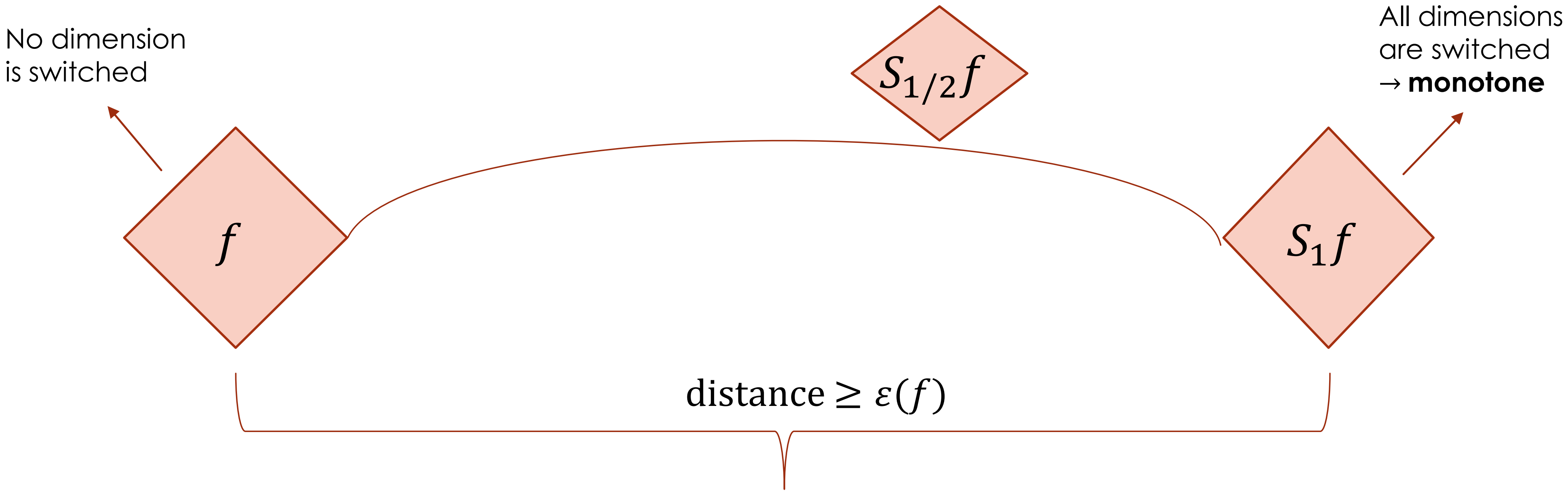


- Then it holds that for $p = \frac{1}{2}$:

$$\varepsilon(g) \geq \mathbf{E}\left[\varepsilon(S_{1/2} f)\right]$$



Phase 3 Idea, part 2



- **Phase 3:** Then it holds that:

$$
\begin{aligned}
\varepsilon(g) &\geq \mathbf{E}\left[\varepsilon(S_{1/2}f)\right] \\
&\approx \mathbf{E}\left[\mathrm{dist}\left(S_{1/2}f,\ S_1 f\right)\right] \\
&\geq \mathbf{E}[\mathrm{dist}(f, S_1 f)] - \mathbf{E}\left[\mathrm{dist}\left(f, S_{1/2}f\right)\right]
\end{aligned}
$$

Actually need expectation over all possible orderings of coordinates

Final step of proof

- For $i = 0, 1, \dots, 5\log n$:

$$\mathbf{E}_x\left[\sqrt{I_f^-(x)}\right] \geq \mathbf{E}[\mathrm{dist}\left(f, S_{1/2^i} f\right)] - \mathbf{E}[\mathrm{dist}\left(f, S_{1/2^{i+1}} f\right)]$$

- Telescoping sum:

For $i = 0$: f is switched in every dimension, expected distance is $\varepsilon(f)$

For $i = 5\log n$: w.h.p f is not switched in any dimension, expected distance is ≈ 0

- Hence:

$$\log n \cdot \mathbf{E}_x\left[\sqrt{I_f^-(x)}\right] \geq \varepsilon(f)$$

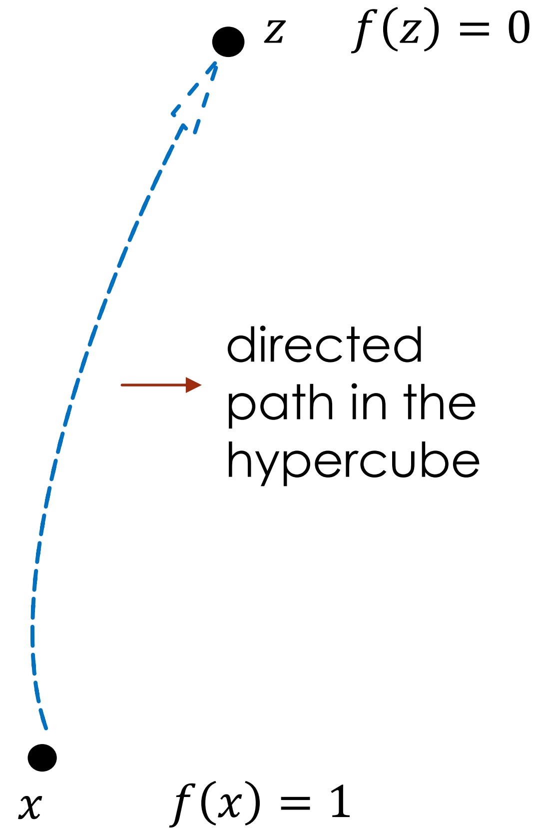
The $\sqrt{n}$ –tester

Given n and ε:

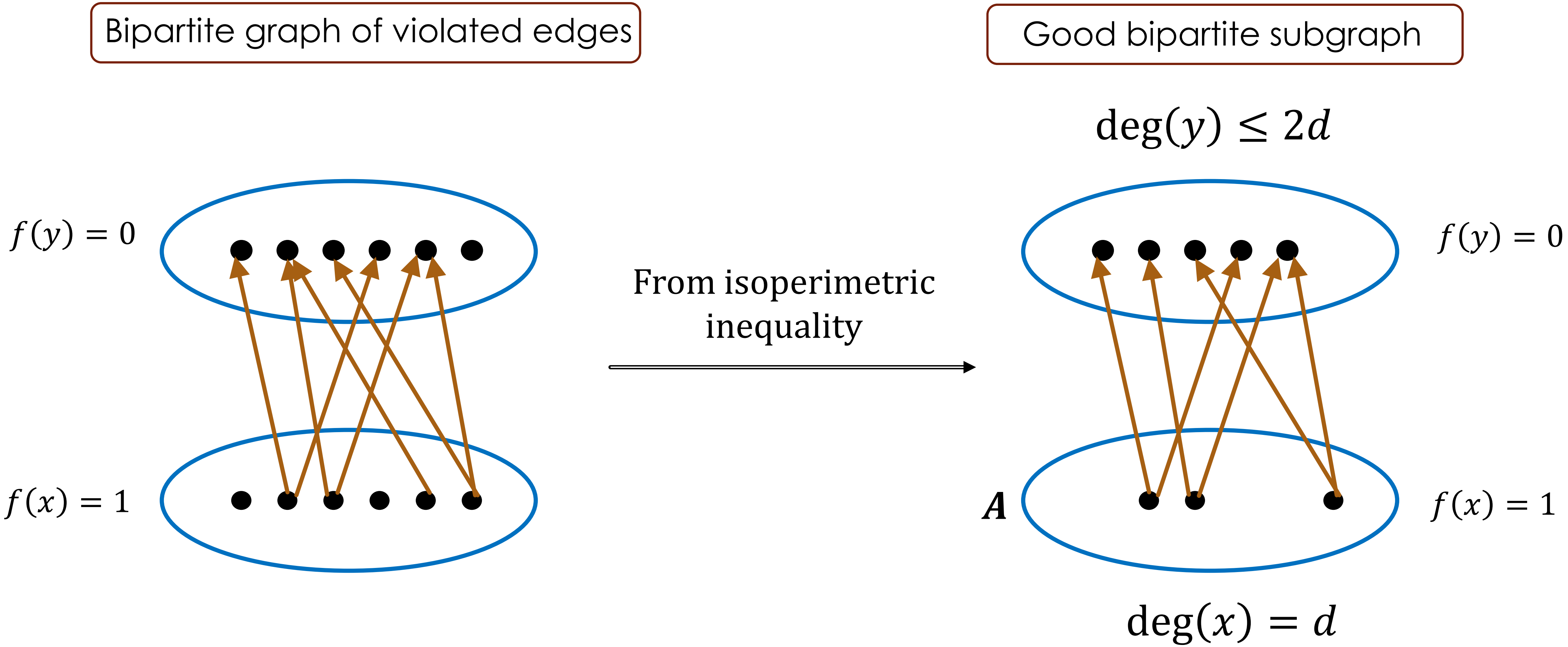
Repeat $\tilde{O}\left(\frac{\sqrt{n}}{\varepsilon^2}\right)$ times:

- Sample x from $\{0,1\}^n$
- Sample k from $\{0,1,2,\ldots,\log\sqrt{n}\}$
- Obtain z by changing 2^k coordinates of x from 0 to 1
- Reject if $f(x) = 1$ and $f(z) = 0$

Accept



Analysis of the $\sqrt{n}$ tester

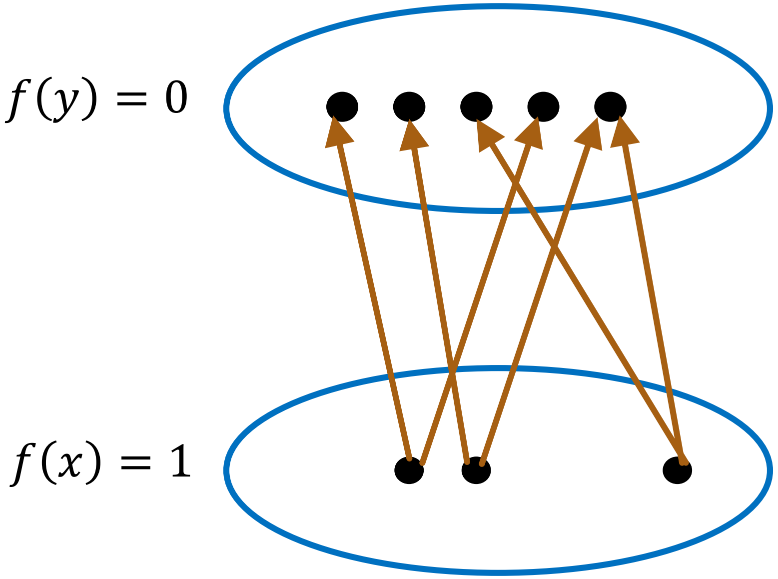


Either A or $\sqrt{d}$ is big! ← $\frac{|A|}{2^n}\sqrt{d} \geq \frac{\varepsilon(f)}{\log n}$

Analysis of the $\sqrt{n}$ tester

Good bipartite subgraph

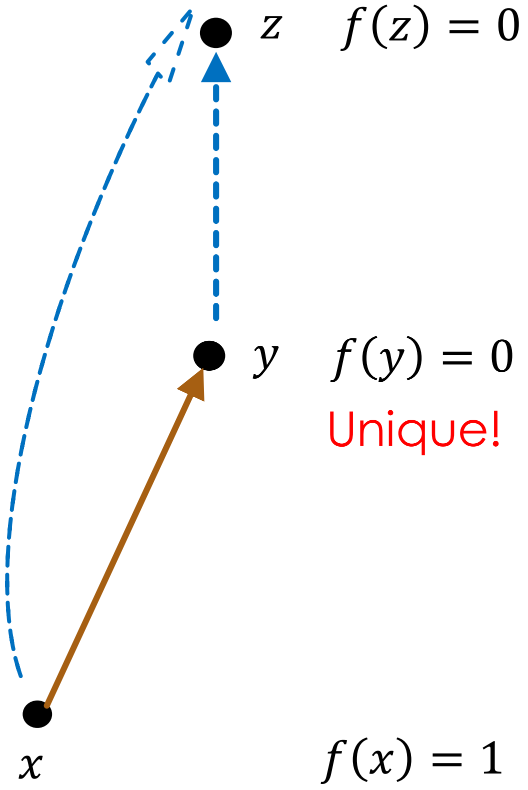
$$\deg(y) \le 2d$$



$$\deg(x) = d$$

$$\frac{|A|}{2^n}\sqrt{d} \ge \frac{\varepsilon(f)}{\log n}$$

With high probability $\Longrightarrow$



Conclusion

- Showed an analysis of the edge tester
- Overview of isoperimetric inequalities
- Outlined proof of main inequality in the $\sqrt{n}$ - tester
- Related isoperimetric inequality to analysis of $\sqrt{n}$ - tester

- **Open problems:**
  - Gap between lower bound and upper bound for Boolean functions on hypergrid
    - $f: [n]^d \rightarrow \{0,1\}$
    - Lower bound is $\Omega(\sqrt{d})$ and upper bound is $\tilde{O}(d^{\frac{5}{6}})$ [Black, Chakrabarty, Seshadri '17]
  - Better adaptive algorithm or better adaptive lower bound
    - Current lower bound: $\tilde{\Omega}\left(n^{\frac{1}{3}}\right)$ [Chen, Waingarten, Xie '17]