Isoperimetric Inequalities For Real-Valued Functions with Applications to Monotonicity Testing

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Overview

For Boolean functions on the hypercube: $f: \{0,1\}^d \to \{0,1\}$.

Undirected

- Margulis '74
- Talagrand '93

Directed

- Chakrabarty and Seshadhri '13
- Khot, Minzer, Safra '15

We generalize these inequalities to **real-valued** functions: $f: \{0,1\}^d \to \mathbb{R}$.

Motivation:

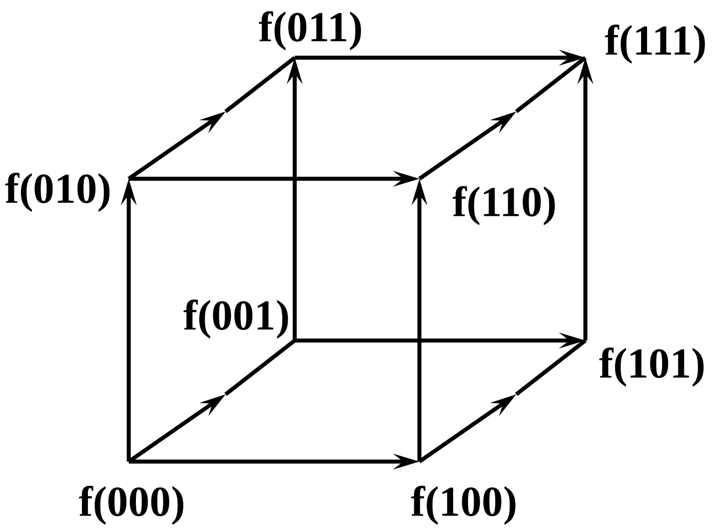
- To understand the structure of real-valued functions.
- To improve sublinear algorithms for monotonicity.

Plan

1. Explain our results in monotonicity testing.
2. Give some background on the inequalities.
3. Prove our generalized inequalities.

The d-dimensional hypercube

- Hypercube has 2^d vertices, the points in $\{0,1\}^d$.



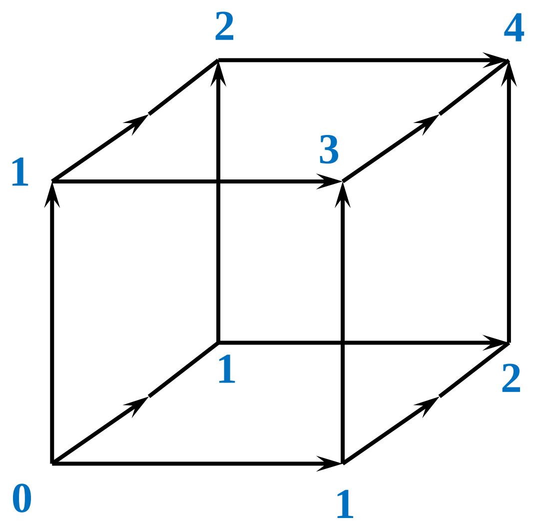
- $x \rightarrow y$ is an edge if:
    - $x_i = 0,\ y_i = 1$
    - $x_j = y_j$ for all $j \in [n] \backslash \{i\}$

- f is monotone if the value of f along every edge does not decrease.

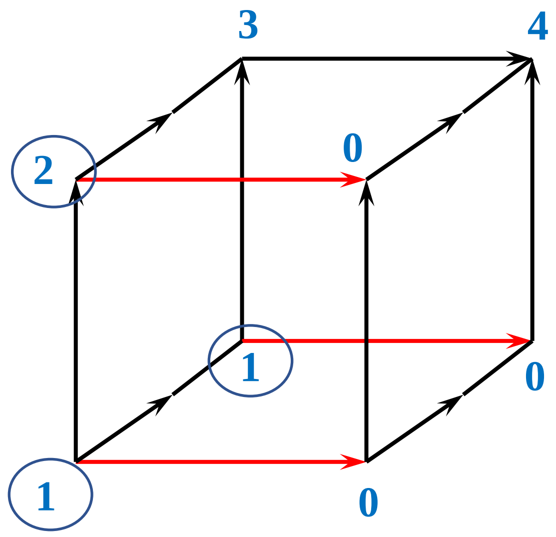
- Edge $x \rightarrow y$ is influential if $f(x) \neq f(y)$.
- Edge $x \rightarrow y$ is violated if $f(x) > f(y)$.

Distance to monotonicity

- Let $\mathbf{dist}(\boldsymbol{f}, \mathbf{mono})$ denote the distance of f to monotonicity
- $\mathbf{dist}(\boldsymbol{f}, \mathbf{mono})$ = least number values of f that need to be changed to make f monotone



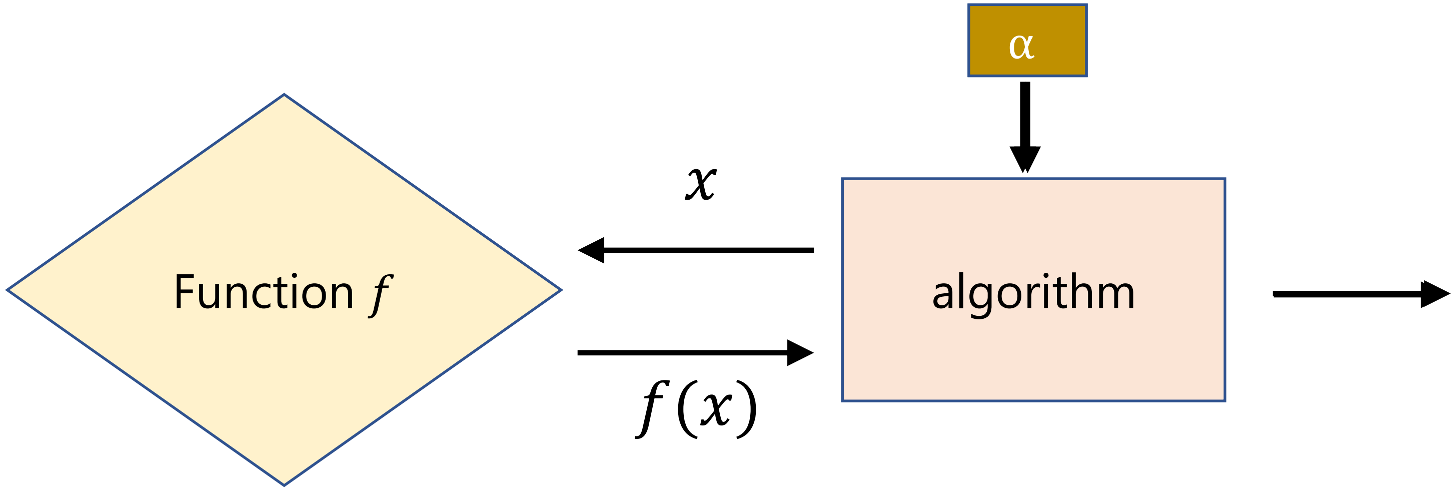
$\mathrm{dist}(f, \mathrm{mono}) = 0$



$\mathrm{dist}(f, \mathrm{mono}) = 3$

Algorithmic tasks

- Monotonicity testing: [Rubinfeld Sudan '96] [Goldreich Goldwasser Ron '98] [Goldreich Goldwasser Lehman Ron Samorodnitsky '00]



- Accept if f monotone
- Reject whp if far from monotone:
  $\mathbf{dist}(\boldsymbol{f}, \mathbf{mono}) \geq \varepsilon \cdot 2^d$

Estimate $\hat{\varepsilon}$

- Approximating distance to monotonicity: [Parnas, Ron, Rubinfeld '06], [Fattal, Ron '10]
  - Given oracle access to f s.t. $\mathrm{dist}(f, \mathrm{mono}) \geq \alpha \cdot 2^d$.
  - Achieves $\boldsymbol{c}$-approximation if it returns estimate $\hat{\varepsilon}$ that whp:

$$\mathrm{dist}(f, \mathrm{mono}) \leq \hat{\varepsilon} \leq \boldsymbol{c} \cdot \mathrm{dist}(f, \mathrm{mono})$$

can turn this into additive error

Results – Monotonicity Testing

Extensively studied problem [Ergun, Kannan, Kumar, Rubinfeld, Viswanathan '00][Dodis Goldreich Lehman Raskhodnikova '99][Lehman Ron '01][Ailon Chazelle '06][Fischer '04][Halevy Kushilevitz '08][Batu Rubinfeld White '05][Ailon Chazelle Seshadhri Liu '07][Bhattacharyya Grigorescu Jung Raskhodnikova Woodruff '12][Briet Chakraborty Soriano Matsliah '12][Blais Raskhodnikova Yaroslavtsev '14][Chakrabarty Seshadhri '13'14'16'19][Chen Servedio Tan '14][Belovs Blais '16][Pallavoor Raskhodnikova Varma '18][Black Chakrabarty Seshadhri '18'20]

Functions on the hypercube $\{0,1\}^d$, $\boldsymbol{r}$ = number of distinct values of f.

	Boolean	Real-Valued (Previous)	Real-Valued (Our results)
Upper bounds	$\tilde{O}\left(\frac{\sqrt{d}}{\varepsilon^2}\right)$ [Khot Minzer Safra '15]	$O\left(\frac{d}{\varepsilon}\right)$ [Chakrabarty Seshadhri '13]	$\tilde{O}\left(\min\left(\frac{r\sqrt{d}}{\varepsilon^2}, \frac{d}{\varepsilon}\right)\right)$
Lower Bounds	Nonadaptive: $\tilde{\Omega}(\sqrt{d})$ [Fischer Lehman Newman Raskhodnikova Rubinfeld '02] [Chen De Servedio Tan '15] [Chen Waingarten Xie '17] Adaptive : $\tilde{\Omega}(\mathrm{d}^{1/3})$ [Chen Waingarten Xie '17]	$\Omega(\min(d, r^2))$ [Blais Brody Matulef '12]	$\Omega(\min(r\sqrt{d}, d))$ Nonadaptive, 1-sided error

nonadaptive = tester makes all its queries in advance

Results – Distance Approximation

Functions on the hypercube: $\{0,1\}^d$, r = number of distinct values of f

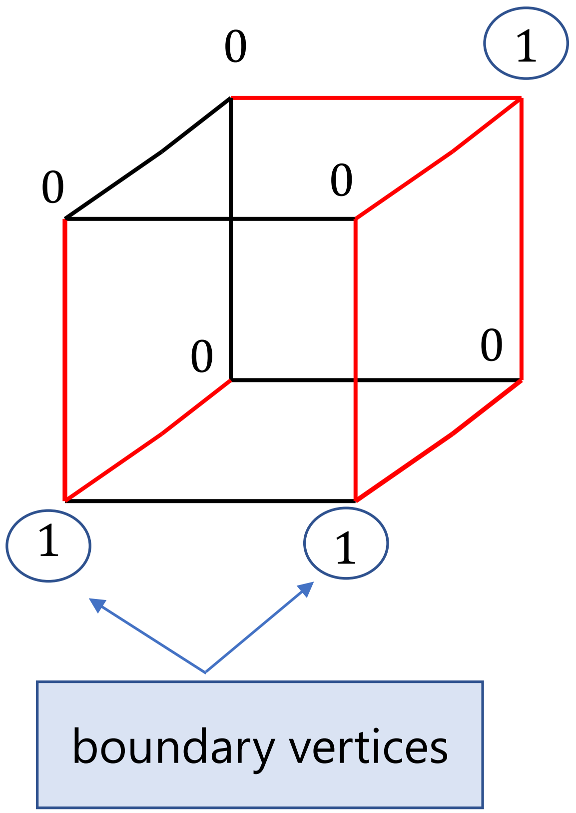
	Boolean	**Real-Valued (Previous)**	**Real-Valued (Our results)**
Upper bounds	$\sqrt{d \log d}$-factor [Pallavoor Raskhodnikova Waingarten '20]	$d \log r$ –factor [Fattal Ron '10]	$\sqrt{d \log d}$-factor
Lower bounds	$\sqrt{d}$ - factor (nonadaptive) [Pallavoor Raskhodnikova Waingarten '20]		

no dependence on r

All algorithms have query complexity $poly(d, \frac{1}{\alpha})$

Isoperimetric Inequalities (Undirected)

- An edge (x, y) is influential if $f(x) \neq f(y)$.
- Let $I_f(x) = $ # influential edges (x, y) s.t. $f(x) > f(y)$.



- [Talagrand '93] For a Boolean function f,

$p_0 =$ fraction of zeros
$\mathrm{var}(f) = p_0(1 - p_0)$

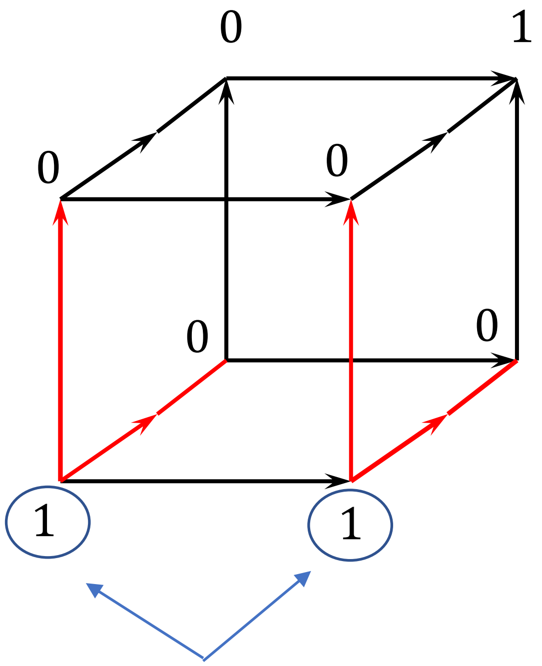
$$\sum_{x \in \{0,1\}^d} \left[\sqrt{I_f(x)}\right] = \Omega\big(\mathrm{var}(f)\big) \cdot 2^d$$

- [Margulis '74] For a Boolean function f,

$$\frac{(\#\text{influential edges}) \cdot (\#\text{boundary vertices})}{2^{2d}} = \Omega(\mathrm{var}(f)^2)$$

Isoperimetric Inequalities (Directed)

- An edge $x \to y$ is violated if $f(x) > f(y)$.
- Let $I_f^-(x)$ = # outgoing violated edges at x.



boundary vertices

- [Khot Minzer Safra '15] For a Boolean function f,
  [Pallavoor Raskhodnikova Waingarten '20]

$$\sum_{x \in \{0,1\}^d} \left[\sqrt{I_f^-(x)} \right] = \Omega\big(\mathrm{dist}(f, \mathrm{mono})\big)$$

- [Chakrabarty Seshadhri '13] For a Boolean function f,

$$(\#\text{violated edges}) \cdot (\#\text{boundary vertices}) \; = \Omega(\mathrm{dist}(f, \mathrm{mono})^2)$$

variance → dist to mono

Our inequalities

- (**Directed**) For all real-valued functions $f\colon \{0,1\}^d \to \mathbb{R}$:

$I_f^-(x)$ = # outgoing violated edges at x

$$\sum_{x \in \{0,1\}^d} \left[\sqrt{I_f^-(x)}\right] = \Omega(\mathrm{dist}(f, \mathrm{mono}))$$

no dependence on the range of f

- (**Undirected**) For all real-valued functions $f\colon \{0,1\}^d \to \mathbb{R}$:

$I_f(x)$ = # influential edges at x

$$\sum_{x \in \{0,1\}^d} \left[\sqrt{I_f(x)}\right] = \Omega(\mathrm{dist}(f, \mathbf{constant}))$$

Number of values that need to be changed to make f constant

For a Boolean function, variance and normalized distance to constant are within a factor of 2

We don't care about the magnitude of change

Main inequality

$I_f^-(x) = \#$ outgoing violated edges at x

- (**Directed**) For all real-valued functions $f\colon \{0,1\}^d \to \mathbb{R}$:

$$\sum_{x\in\{0,1\}^d} \left[\sqrt{I_f^-(x)}\right] = \Omega\big(\mathrm{dist}(f,\mathrm{mono})\big)$$

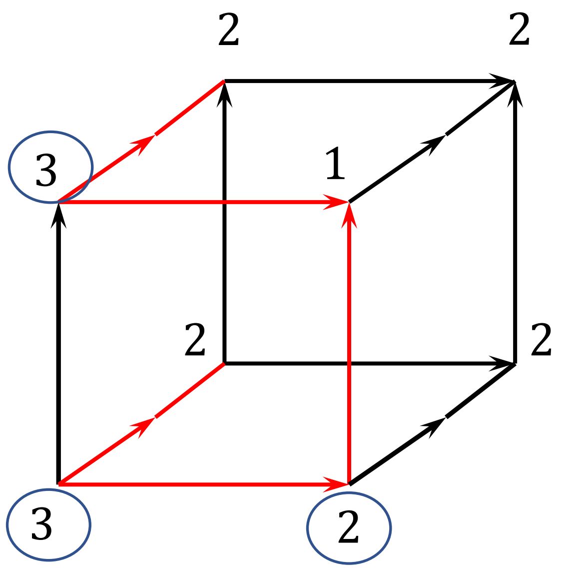
- Inequality we use for our applications.
- Implies all other inequalities mentioned in this talk.
- We show how to prove it.

Main inequality

$I_f^-(x)$ = # outgoing violated edges at x

- (**Directed**) For all real-valued functions f: $\{0,1\}^d \to \mathbb{R}$:

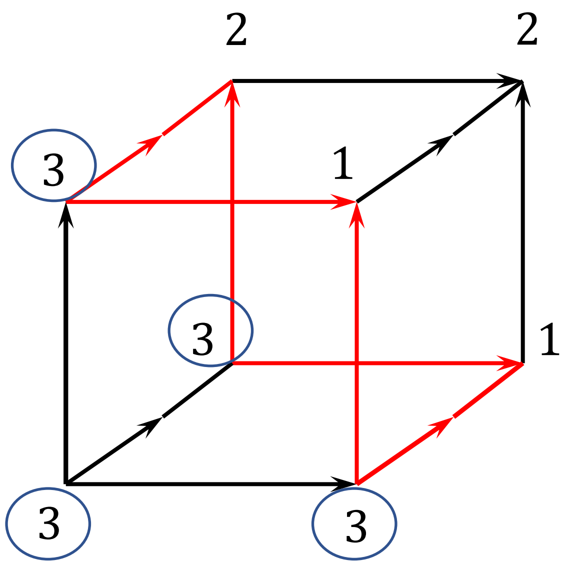
$$\sum_{x\in\{0,1\}^d}\left[\sqrt{I_f^-(x)}\right] = \Omega\big(\mathrm{dist}(f,\mathrm{mono})\big)$$



$\mathrm{dist}(f,\mathrm{mono}) = 3$

$$\sum_{x\sim\{0,1\}^d}\left[\sqrt{I_f^-(x)}\right]$$

$$= \sqrt{2} + \sqrt{2} + \sqrt{1}$$



$\mathrm{dist}(f,\mathrm{mono}) = 4$

$$\sum_{x\sim\{0,1\}^d}\left[\sqrt{I_f^-(x)}\right]$$

$$= \sqrt{2} + \sqrt{2} + \sqrt{2}$$

Main inequality

$I_f^-(x)$ = # outgoing violated edges at x

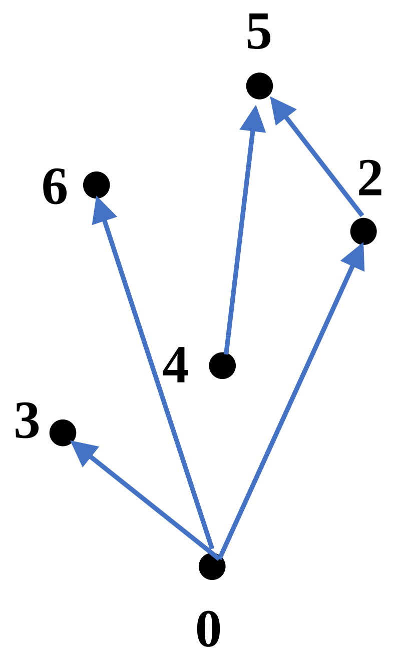
- (**Directed**) For all real-valued functions f: $\{0,1\}^d \to \mathbb{R}$:

$$\sum_{x\in\{0,1\}^d} \left[\sqrt{I_f^-(x)}\right] = \Omega\big(\mathrm{dist}(f,\mathrm{mono})\big)$$

- We prove it by reducing to the Boolean case, via Boolean Decomposition Theorem.

Boolean Decomposition Theorem

- It works for every partially ordered domain, which we represent as a DAG G.
- Monotonicity testing on posets first considered by [Fischer Lehman Newman Raskhodnikova Rubinfeld '02].
- Vertices $V(G)$, edges $E(G)$.
- $x \preccurlyeq y$ iff there is directed path from x to y.
- Edge $x \rightarrow y$ is violated if $f(x) > f(y)$.



$f : V(G) \rightarrow \mathbb{R}$

Boolean Decomposition Theorem

- Let $\mathrm{VIOL}(f)$ denote the violated edges of f.

BD Theorem: Let G be a DAG, and $f : V(G) \to \mathbb{R}$ a nonmonotone function. For some $k \geq 1$, there exist Boolean functions $f_1, f_2, \ldots, f_k : V(G) \to \{0,1\}$ and disjoint subgraphs $H_1, H_2, \ldots, H_k$ of G such that:

(1) $\sum_{i \in [k]} \mathrm{dist}(f_i, \mathrm{mono}) \geq \frac{1}{2} \mathrm{dist}(f, \mathrm{mono})$

collectively capture distance to monotonicity of f

(2) $\mathrm{VIOL}(f_i) \subseteq \mathrm{VIOL}(f)$

edges violated by f_i are also violated by f

(3) $\mathrm{VIOL}(f_i) \subseteq E(H_i)$

edges violated by f_i are contained in H_i

BD Theorem → Main inequality

$I_f^-(x)$ = # outgoing violated edges at x

$$\sum_{x\in\{0,1\}^d}\left[\sqrt{I_f^-(x)}\right] \geq \sum_{x\in\cup H_i}\left[\sqrt{I_f^-(x)}\right]$$

$\cup H_i$ is a subgraph of original graph

$$\geq \sum_{i\in[k]}\sum_{x\in H_i}\left[\sqrt{I_f^-(x)}\right]$$

the H_i are disjoint subgraphs

edges violated by f_i are in H_i

$$\geq \sum_{i\in[k]}\sum_{x\in H_i}\left[\sqrt{I_{f_i}^-(x)}\right]$$

edges violated by f_i are a subset of edges violated by f

from the Boolean case

$$\geq \sum_{i\in[k]} C\cdot \mathrm{dist}(f_i, \mathrm{mono}) \geq \frac{C}{2}\cdot \mathrm{dist}(f, \mathrm{mono})$$

capture $\mathrm{dist}(f, \mathrm{mono})$

Proof of BD Theorem

BD Theorem: Let G be a DAG, and $f: V(G) \to \mathbb{R}$ a nonmonotone function. For some $k \geq 1$, there exist Boolean functions $f_1, f_2, \dots, f_k: V(G) \to \{0,1\}$ and disjoint subgraphs $H_1, H_2, \dots, H_k$ of G such that:

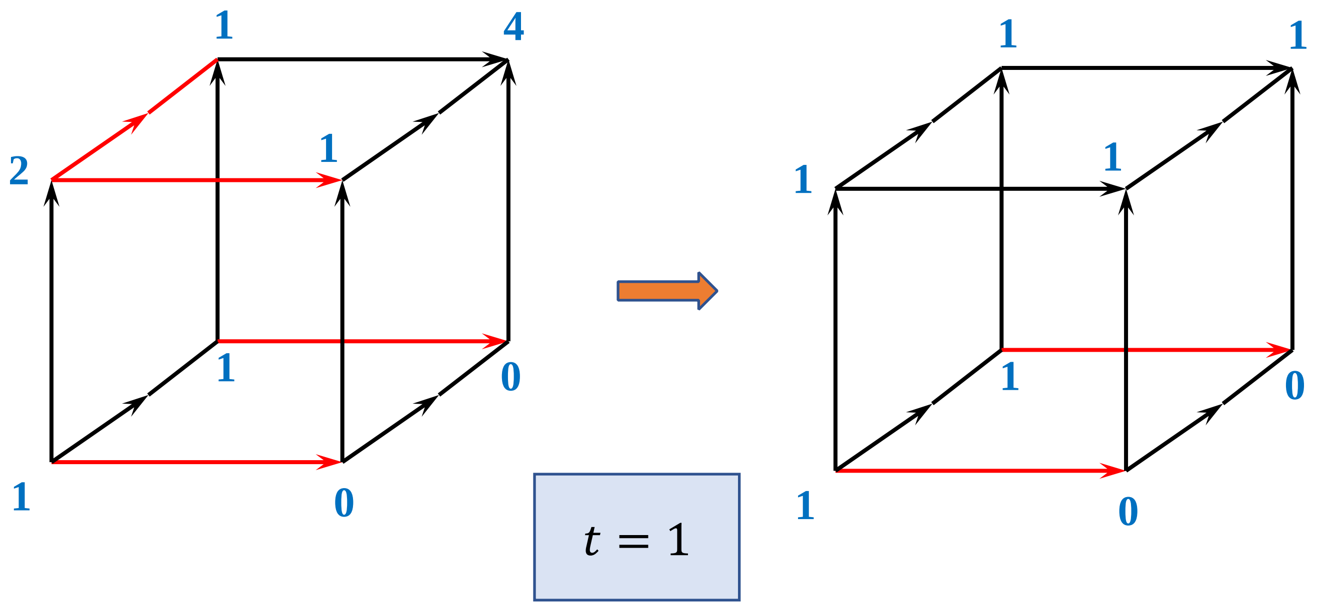
(1) $\sum_{i \in [k]} \mathrm{dist}(f_i, \mathrm{mono}) \geq \frac{\mathrm{dist}(f, \mathrm{mono})}{2}$ (2) $\mathrm{VIOL}(f_i) \subseteq \mathrm{VIOL}(f)$ (3) $\mathrm{VIOL}(f_i) \subseteq E(H_i)$

$\Rightarrow$ Main inequality $\sum_{x \in \{0,1\}^d} \left[\sqrt{I_f^-(x)}\right] = \Omega\big(\mathrm{dist}(f, \mathrm{mono})\big)$

Thresholding intuition

We can reduce from real-valued to Boolean functions via thresholding.

$$h_t(x) = \begin{cases} 1 & \text{if } f(x) \geq t \\ 0 & \text{if } f(x) < t \end{cases}$$



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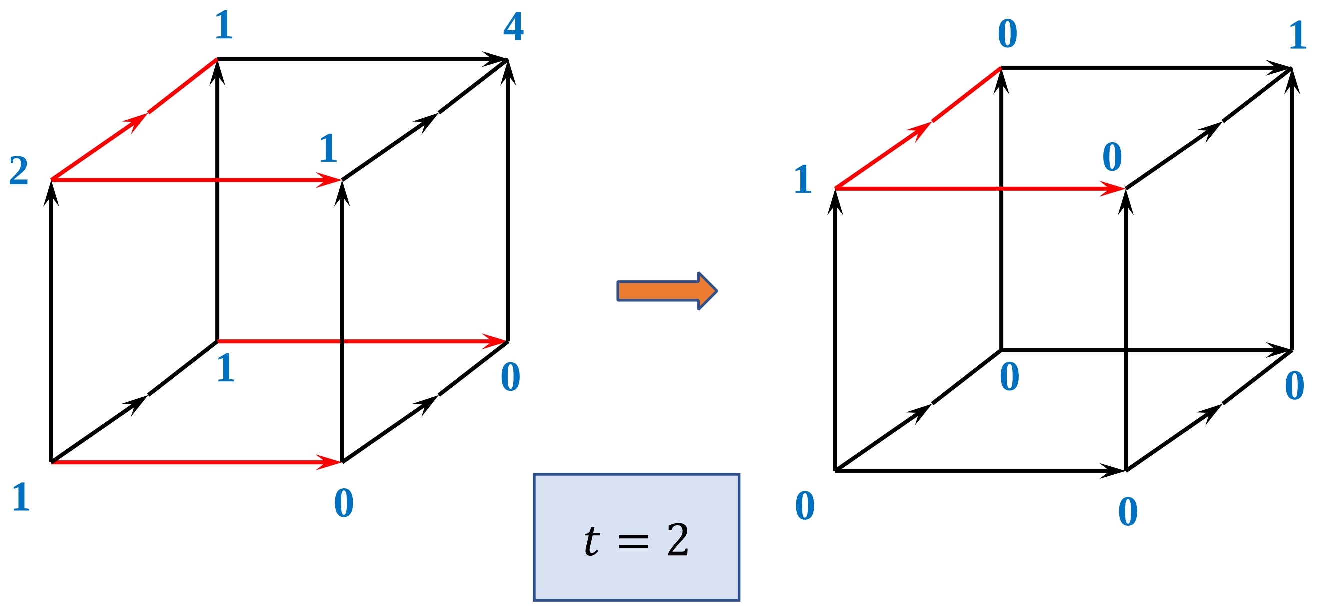


- Edges violated by h_t are a subset of the edges violated by f.
- But $\mathrm{dist}(f, \mathrm{mono})$ can decrease by a factor of r (# distinct values of f)
- Can construct function so that $\mathrm{dist}(f, \mathrm{mono})$ decreases by r for all thresholds $t \in [r]$.
- BD Theorem allows us to apply different thresholds in disjoint locations of hypercube.

Proof of BD Theorem

BD Theorem: Let G be a DAG, and $f: V(G) \to \mathbb{R}$ a nonmonotone function. For some $k \geq 1$, there exist Boolean functions $f_1, f_2, \dots, f_k: V(G) \to \{0,1\}$ and disjoint subgraphs $H_1, H_2, \dots, H_k$ of G such that:

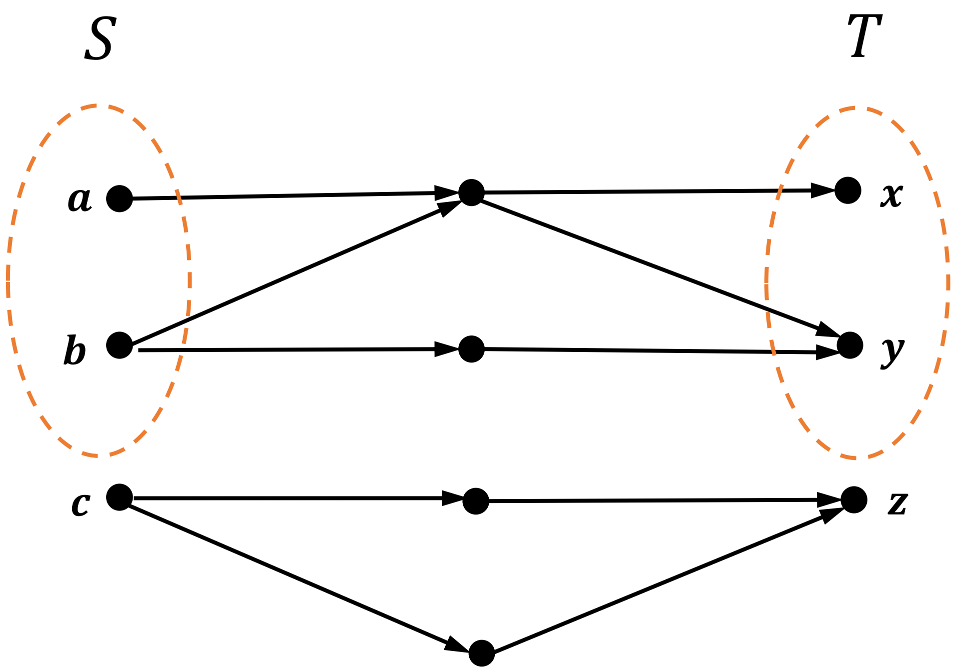
(1) $\sum_{i \in [k]} \mathrm{dist}(f_i, \mathrm{mono}) \geq \frac{\mathrm{dist}(f, \mathrm{mono})}{2}$ (2) $\mathrm{VIOL}(f_i) \subseteq \mathrm{VIOL}(f)$ (3) $\mathrm{VIOL}(f_i) \subseteq E(H_i)$

1. How to obtain disjoint subgraphs H_i from a matching of vertices.
2. Specify a special matching.
3. Define Boolean functions f_i given subgraphs H_i.
4. Prove desired properties of f_i.

Step 1: Disjoint Subgraphs H_i

Definition (**Sweeping Graphs**) For two disjoint sets of vertices $S, T \subseteq V(G)$:

subgraph $\mathrm{Sweep}(S, T)$ = subgraph formed from union of all directed paths from vertices in S to vertices in T



Call (S, T) a set-pair.

Step 1: Disjoint Subgraphs H_i

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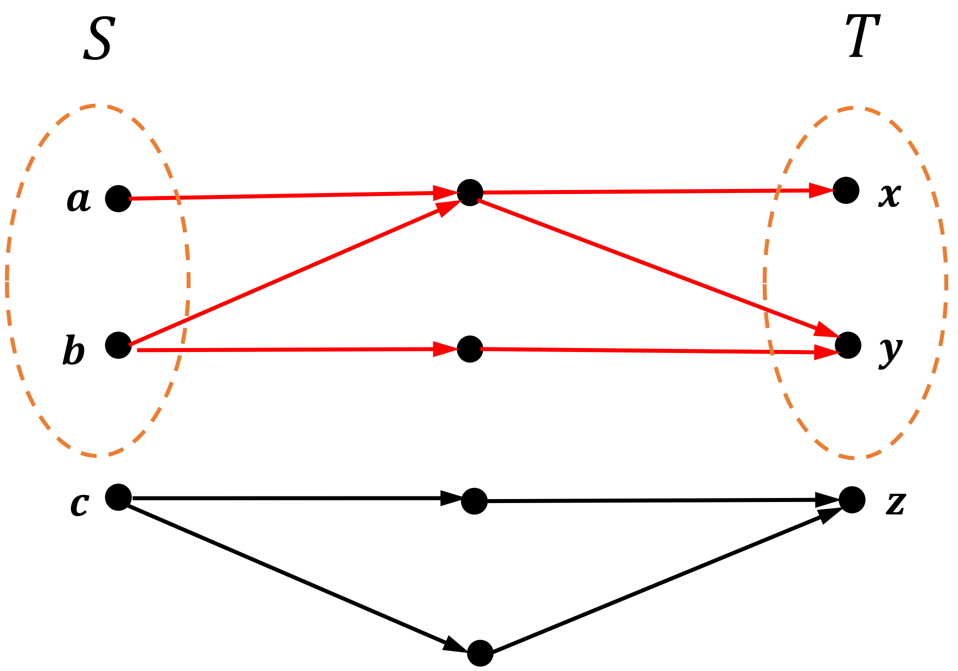


Useful properties:

- $\text{Sweep}(S, T)$ is an induced subgraph
- A vertex outside $\text{Sweep}(S, T)$ cannot be both "above" and "below" $\text{Sweep}(S, T)$

it has a path from a vertex in in $\text{Sweep}(S, T)$

it has a path to a vertex in $\text{Sweep}(S, T)$

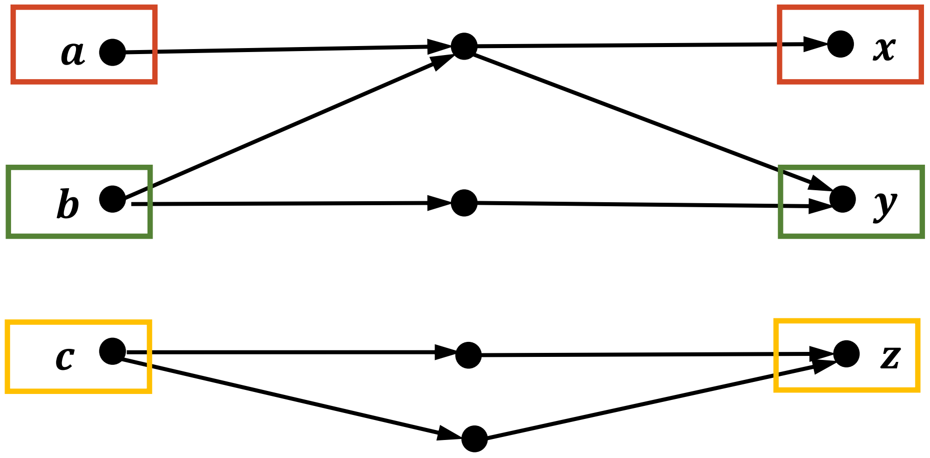
Step 1: Disjoint Subgraphs H_i

We consider matchings $M: S \to T$, where $S, T \subseteq V(G)$.

S = lower endpoints,
T = upper endpoints

M contains disjoint pairs (x, y) of vertices such that $x \preccurlyeq y$.

A pair (x, y) in M is violated if $f(x) > f(y)$.



[Fischer Lehman Newman Raskhodnikova Rubinfeld '02].

Fact. For every function f and maximal matching M of violated pairs:

$$|\text{maximal matching}| \leq \text{dist}(f, \text{mono}) \leq 2|\text{maximal matching}|$$

Step 1: Disjoint Subgraphs H_i

Recall $\mathrm{Sweep}(X, Y)$ = subgraph of paths from vertices in X to vertices in Y

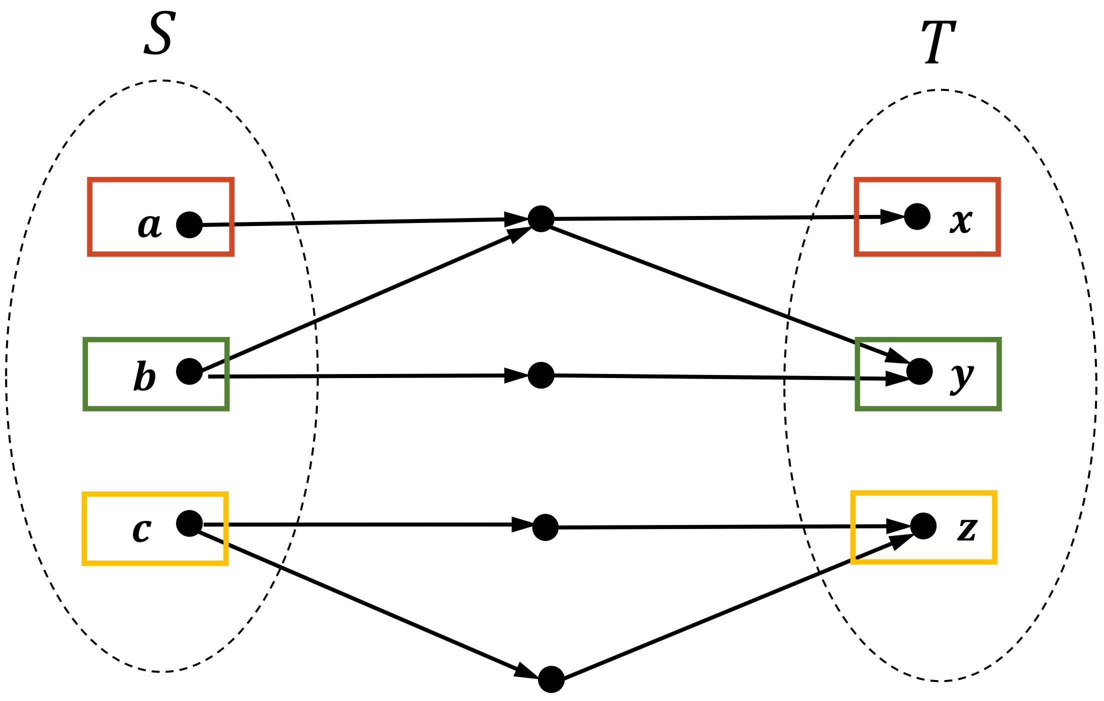
Two set-pairs of vertices (X, Y) and (X', Y') conflict if:

- $\mathrm{Sweep}(X, Y)$ intersects $\mathrm{Sweep}(X', Y')$.

Algorithm Merge-Conflicts:

- Input: matching $M: S \to T$
- Initialize collection of set-pairs $(\{s\}, \{t\})$ for all $(s, t) \in M$
- Repeat until there are no conflicts:
  - if two set-pairs pairs (X, Y) and (X', Y') conflict, merge them,
  - i.e. remove them from collection of pairs, and add new pair $(X \cup X', Y \cup Y')$

Merge-Conflicts Illustration

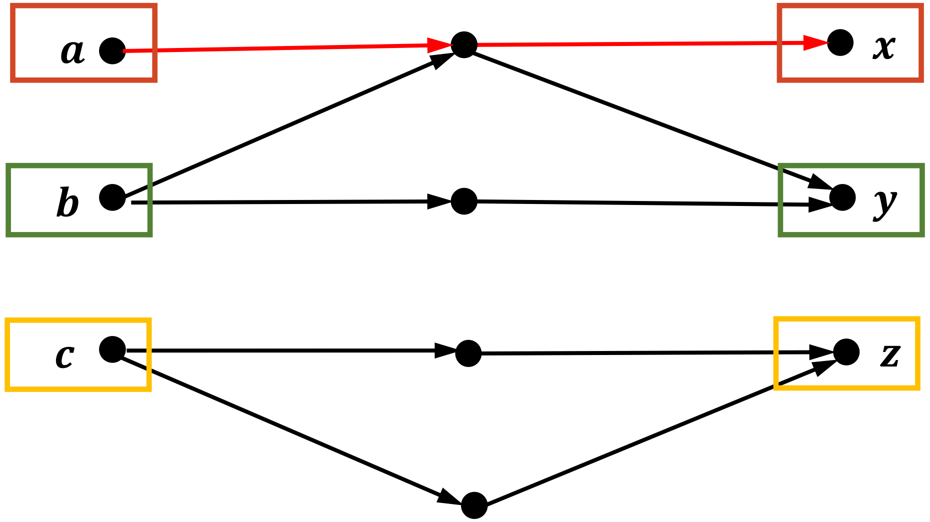


$$M = \{(a, x), (b, y), (c, z)\}$$

Collection = $(\{a\}, \{x\}), (\{b\}, \{y\}), (\{c\}, \{z\})$

Merge-Conflicts Illustration

Sweep$(\{a\}, \{x\})$
union of paths from $\{a\}$ *to* $\{x\}$



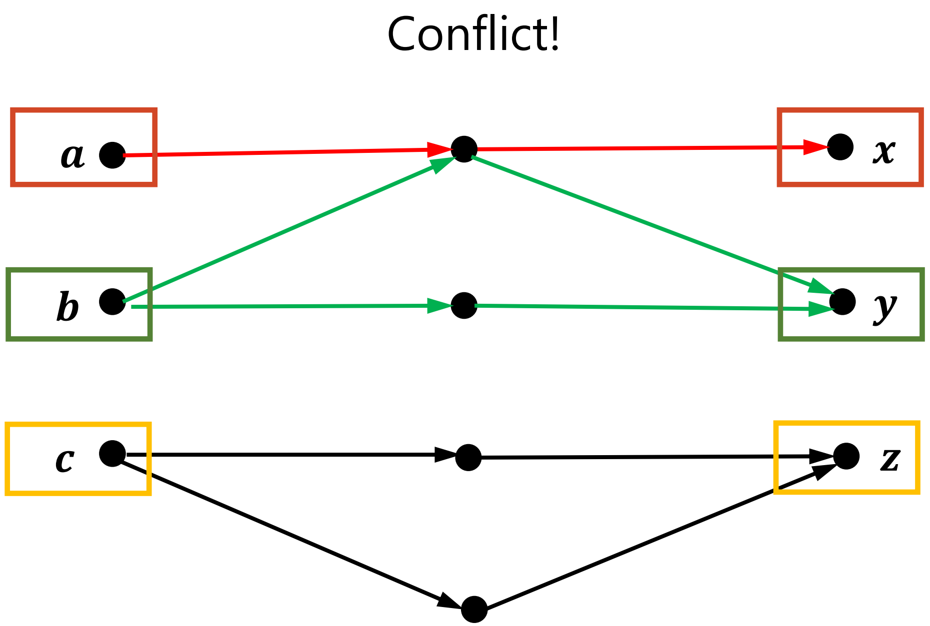
$$M = \{(a, x), (b, y), (c, z)\}$$

Collection = $(\{a\}, \{x\}), (\{b\}, \{y\}), (\{c\}, \{z\})$

Merge-Conflicts Illustration

Sweep$(\{a\},\{x\})$
union of paths from $\{a\}$ *to* $\{x\}$

Sweep$(\{b\},\{y\})$
union of paths from $\{b\}$ *to* $\{y\}$

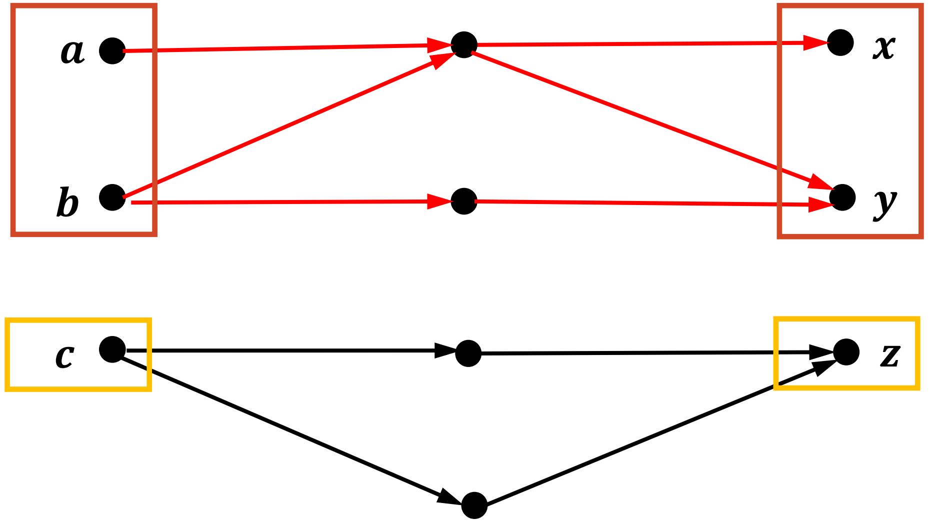


$$M = \{(a,x),(b,y),(c,z)\}$$

Collection = $(\{a\},\{x\}),(\{b\},\{y\}),(\{c\},\{z\})$

Merge-Conflicts Illustration

$\text{Sweep}(\{a,b\},\{x,y\})$
union of paths from $\{a,b\}$ *to* $\{x,y\}$



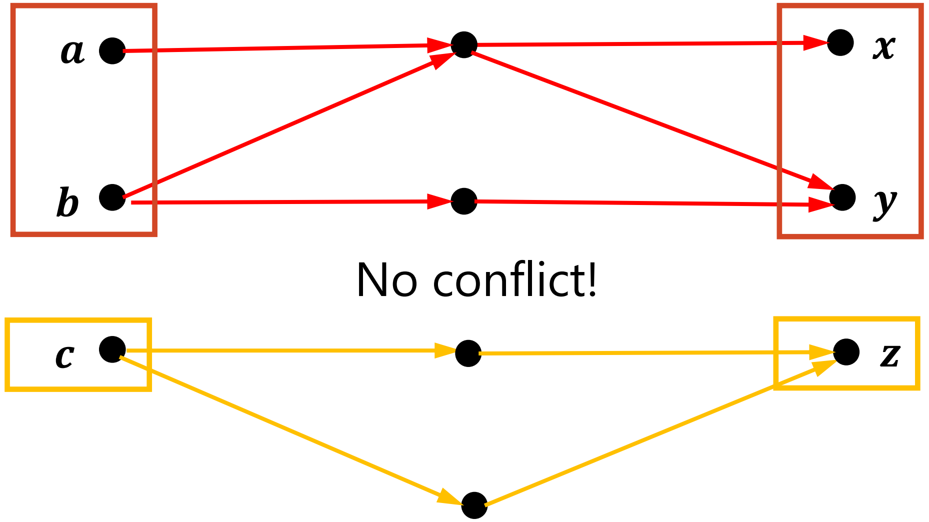
$$M = \{(a,x),(b,y),(c,z)\}$$

Collection $=$ $(\{a,b\},\{x,y\}),(\{c\},\{z\})$

Merge-Conflicts Illustration

Sweep$(\{a, b\}, \{x, y\})$
union of paths from $\{a, b\}$ *to* $\{x, y\}$

Sweep$(\{c\}, \{z\})$
union of paths from $\{c\}$ *to* $\{z\}$



$$M = \{(a, x), (b, y), (c, z)\}$$

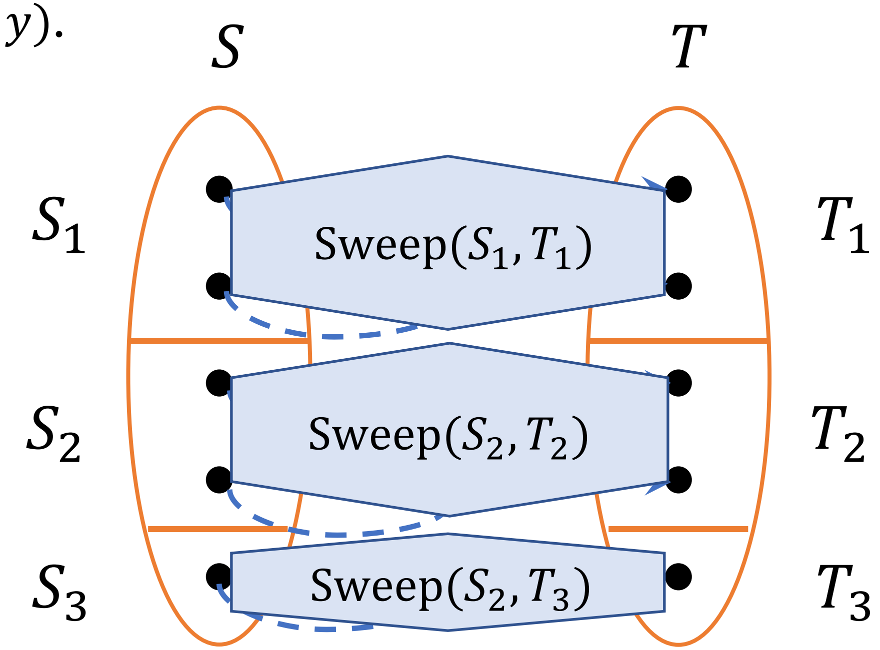
Collection = $(\{a, b\}, \{x, y\}), (\{c\}, \{z\})$

Final collection

Step 1: Disjoint Subgraphs H_i

Algorithm Merge-Conflicts with matching $M: S \to T$ gives set-pairs $(S_1, T_1), (S_2, T_2), \ldots, (S_k, T_k)$ such that:

- The sets S_i partition S, the sets T_i partition T.
- The subgraphs $\mathrm{Sweep}(S_i, T_i)$ are vertex-disjoint.
- **(Rematching property)** For $x \in S_i$, $y \in T_i$ such that $x \preccurlyeq y$: there exists another matching $M': S \to T$ that matches (x, y).



Step 1: Disjoint Subgraphs H_i

Algorithm Merge-Conflicts with matching $M: S \to T$ gives set-pairs $(S_1, T_1), (S_2, T_2), \dots, (S_k, T_k)$ such that:

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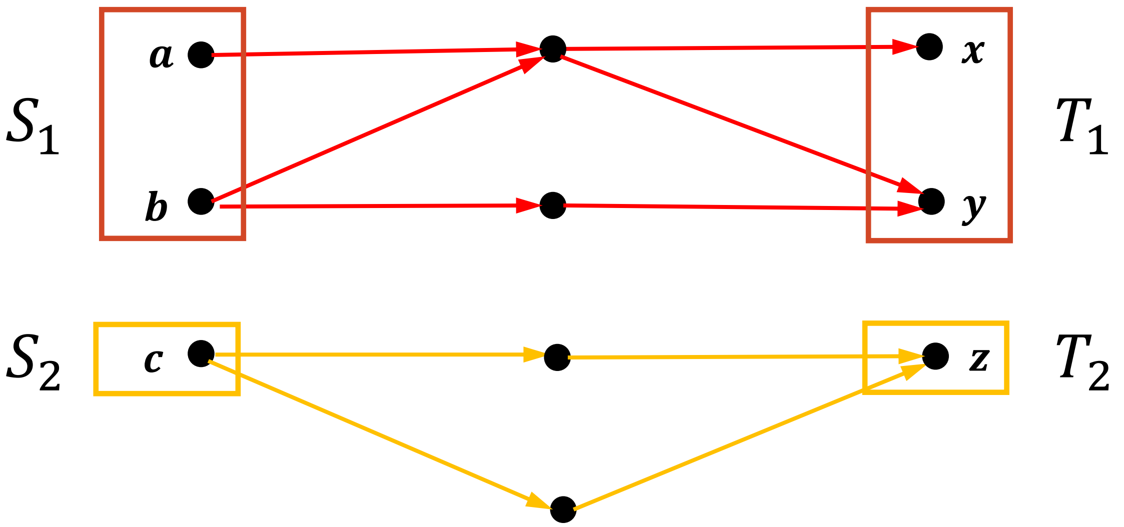


$M = \{(a, x), (b, y), (c, z)\}$ $\Rightarrow$ $M' = \{(a, y), (b, x), (c, z)\}$

Proof of BD Theorem

BD Theorem: Let G be a DAG, and $f: V(G) \to \mathbb{R}$ a nonmonotone function. For some $k \geq 1$, there exist Boolean functions $f_1, f_2, \ldots, f_k: V(G) \to \{0,1\}$ and disjoint subgraphs $H_1, H_2, \ldots, H_k$ of G such that:

(1) $\sum_{i \in [k]} \mathrm{dist}(f_i, \mathrm{mono}) \geq \frac{\mathrm{dist}(f, \mathrm{mono})}{2}$ (2) $\mathrm{VIOL}(f_i) \subseteq \mathrm{VIOL}(f)$ (3) $\mathrm{VIOL}(f_i) \subseteq E(H_i)$

- ✓ How to obtain disjoint subgraphs H_i from a matching of vertices.
2. Specify a special matching.
3. Define Boolean functions f_i given subgraphs H_i.
4. Prove desired properties of f_i.

Step 2: Special matching

Use a special matching M (max-weight, min-cardinality):

- it maximizes **weight** $\sum_{(x,y)\in M}(f(x) - f(y))$,
- and amongst such matchings has the fewest pairs.

- M is maximal
- all pairs in M are violated

Run algorithm Merge-Conflicts with special matching M.

$\text{Sweep}(S_i, T_i)$ are the subgraphs H_i.

Violation Lemma. The set-pairs $(S_1, T_1), (S_2, T_2), \ldots, (S_k, T_k)$ satisfy:

- For all $i \in [k]$, $x \in S_i$, $y \in T_i$, such that $\boxed{x \preccurlyeq y,}$ we have $f(x) > f(y)$.

will need to be more careful about thresholding

can threshold while preserving violations.

Step 2: Special matching

Violation Lemma. The set-pairs $(S_1, T_1), (S_2, T_2), \dots, (S_k, T_k)$ obtained from the special matching M satisfy: For all $i \in [k]$, $x \in S_i$, $y \in T_i$, s.t. $x \preccurlyeq y$, we have $f(x) > f(y)$.

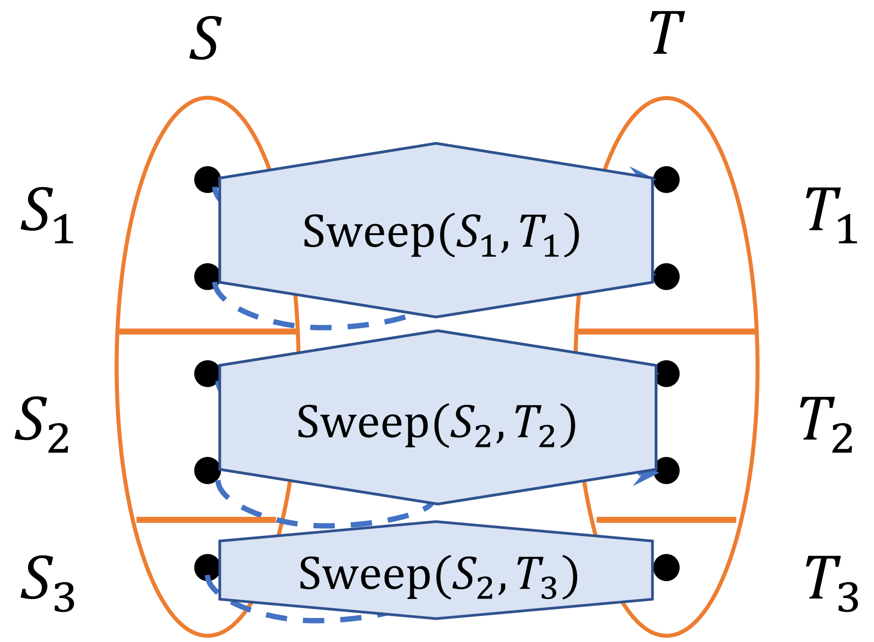
weight: $\sum_{(x,y)\in M}(f(x) - f(y))$

Proof.

- Suppose that for some $x \in S_i$, $y \in T_i$, with $x \preccurlyeq y$ we have $f(x) \le f(y)$.
- Use the rematching property to get a new matching M': $S \to T$ that matches (x, y).
- M' has the same weight as M, since the endpoints have not changed.
- $M' \setminus (x, y)$ has weight at least as big as M, because $f(x) - f(y) \le 0$.
- But $M' \setminus (x, y)$ has fewer pairs. Contradiction.

Step 1+2 summary

- Start with special matching $M: S \to T$ (max weight, min-cardinality).
- M is a maximal matching of violated pairs: $|M| < \mathrm{dist}(f, \mathrm{mono}) < 2|M|$
- Run algorithm Merge-Conflicts to obtain set-pairs $(S_1, T_1), (S_2, T_2), \dots, (S_k, T_k)$
- The subgraphs $\mathrm{Sweep}(S_i, T_i)$ are vertex-disjoint.
- **(Violation Lemma)** For $x \in S_i,\ y \in T_i$ such that $x \preccurlyeq y$ we have $f(x) > f(y)$.



Proof of BD Theorem

BD Theorem: Let G be a DAG, and $f: V(G) \to \mathbb{R}$ a nonmonotone function. For some $k \geq 1$, there exist Boolean functions $f_1, f_2, \dots, f_k: V(G) \to \{0,1\}$ and disjoint subgraphs $H_1, H_2, \dots, H_k$ of G such that:

(1) $\sum_{i \in [k]} \mathrm{dist}(f_i, \mathrm{mono}) \geq \frac{\mathrm{dist}(f, \mathrm{mono})}{2}$ (2) $\mathrm{VIOL}(f_i) \subseteq \mathrm{VIOL}(f)$ (3) $\mathrm{VIOL}(f_i) \subseteq E(H_i)$

- ✓ How to obtain disjoint subgraphs H_i from a matching of vertices.
- ✓ Specify a special matching.

3. Define Boolean functions f_i given subgraphs H_i.
4. Prove desired properties of f_i.

Step 3: Define Boolean Functions

Given (S_i, T_i), define f_i: $V(G) \to \{0,1\}$

max value of f achieved by points in T_i above z

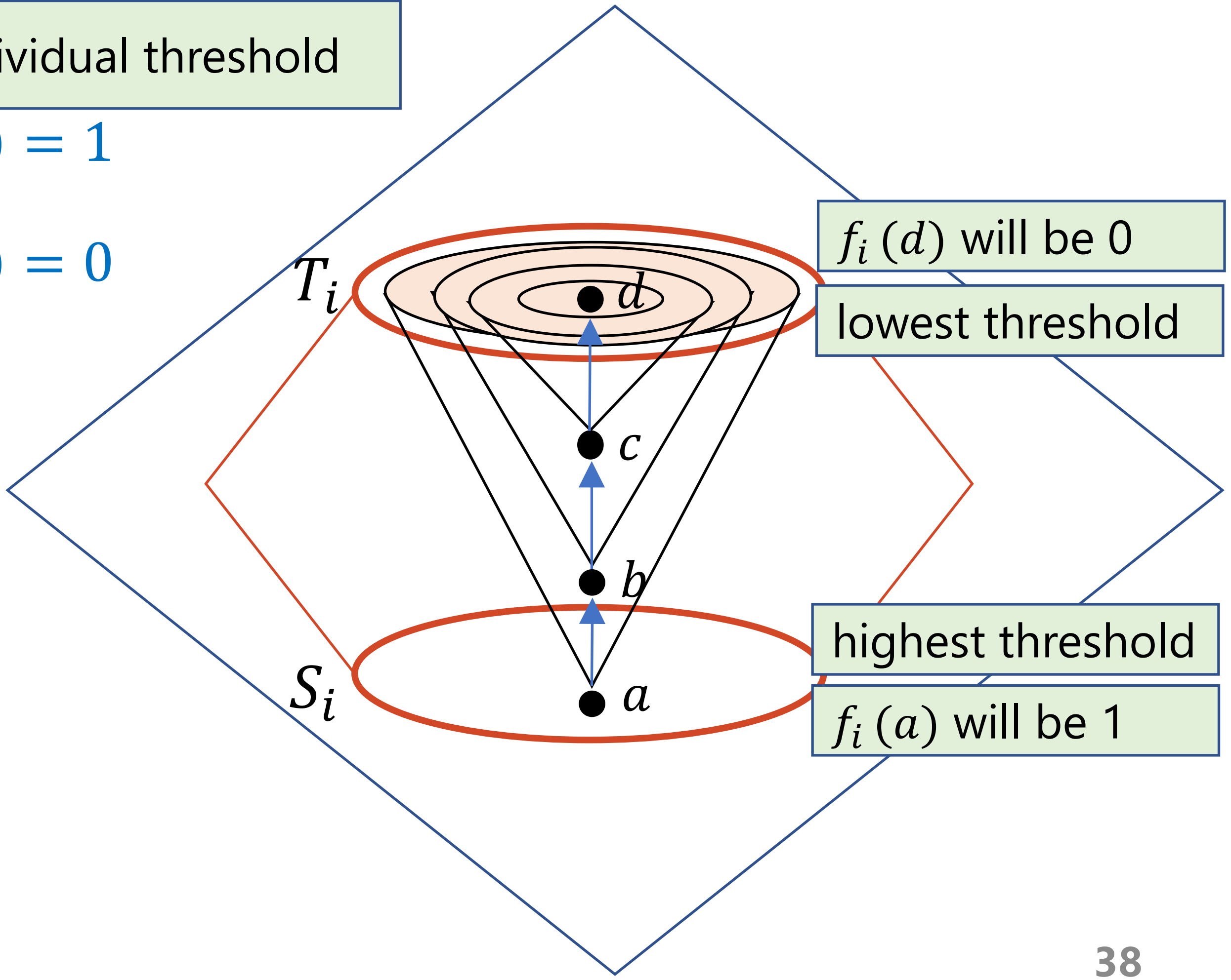
individual threshold

z

in Sweep(S_i, T_i)

- $f(z) > \max_{x \in T_i, z \preccurlyeq x} f(x)$, then $f_i(z) = 1$
- $f(z) \leq \max_{x \in T_i, z \preccurlyeq x} f(x)$, then $f_i(z) = 0$

not in Sweep(S_i, T_i)



Step 3: Define Boolean Functions

Given (S_i, T_i), define f_i: $V(G) \to \{0,1\}$

z

in $\text{Sweep}(S_i, T_i)$

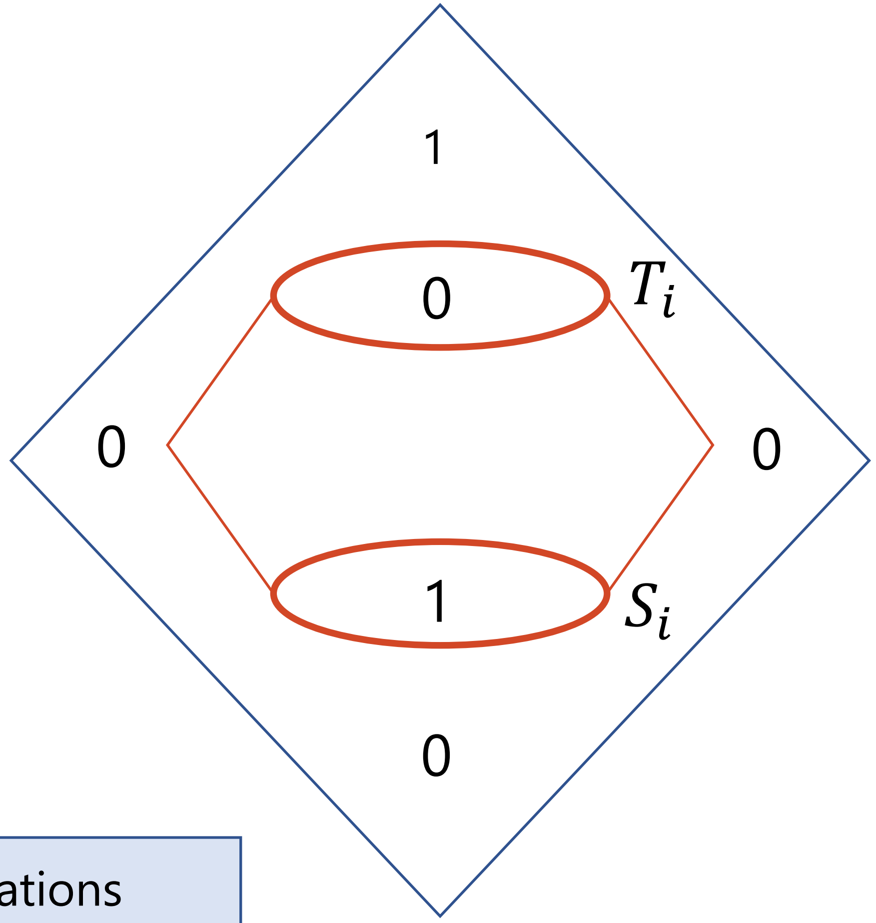
- $f(z) > \max_{x \in T_i, z \preccurlyeq x} f(x)$, then $f_i(z) = 1$
- $f(z) \le \max_{x \in T_i, z \preccurlyeq x} f(x)$, then $f_i(z) = 0$

not in $\text{Sweep}(S_i, T_i)$

- above, then $f_i(z) = 1$
- not above, then $f_i(z) = 0$

a vertex cannot be both above and below $\text{Sweep}(S_i, T_i)$

don't want violations outside of $\text{Sweep}(S_i, T_i)$



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- ✓ How to obtain disjoint subgraphs H_i from a matching of vertices.
- ✓ Specify a special matching.
- ✓ <u>Define Boolean functions f_i given subgraphs H_i</u>.

4. Prove desired properties of f_i.

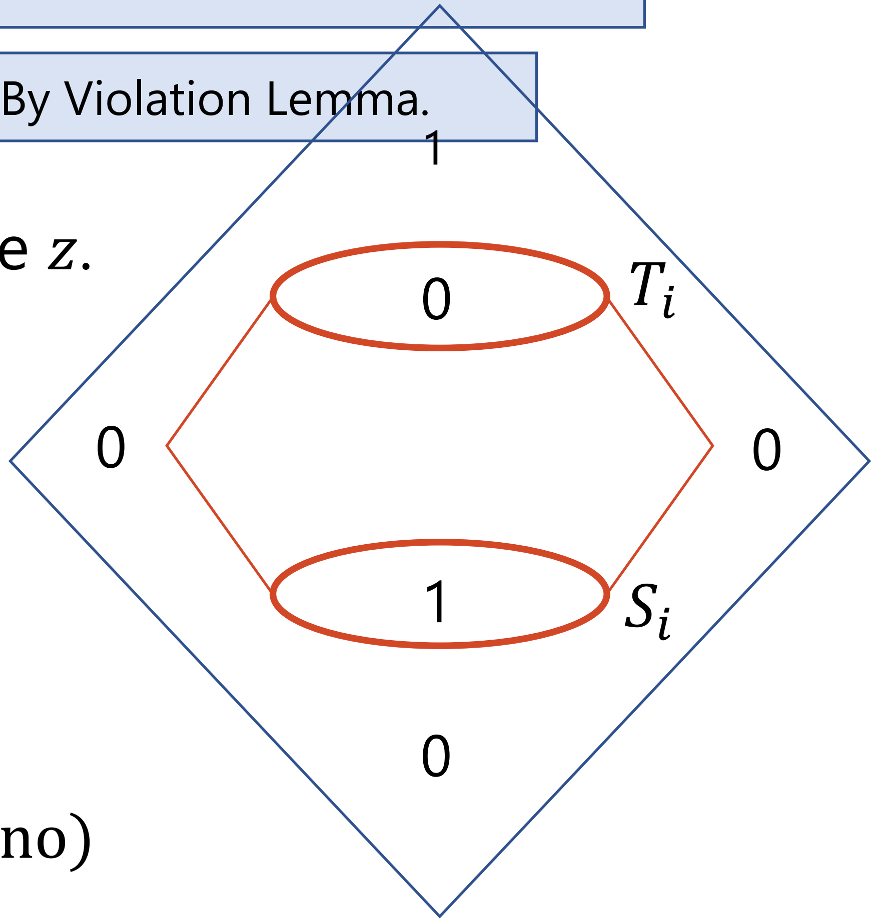
Step 4: Proof of BD Theorem

(1) The functions f_i preserve $\mathrm{dist}(f, \mathrm{mono})$.

- Each vertex in T_i will have value 0.
- Each vertex in S_i will have value 1.
  - If $z \in S_i$, then $f(z) > f(x)$ for all $x \in T_i$ above z.
- $\Rightarrow f_i$ has matching of violated pairs M_i: $S_i \rightarrow T_i$.
- M_i is restriction of M to $\mathrm{Sweep}(S_i, T_i)$.
- All the M_i for $i \in [k]$ are disjoint.

Vertex $z \in T_i$, and z above z

By Violation Lemma.



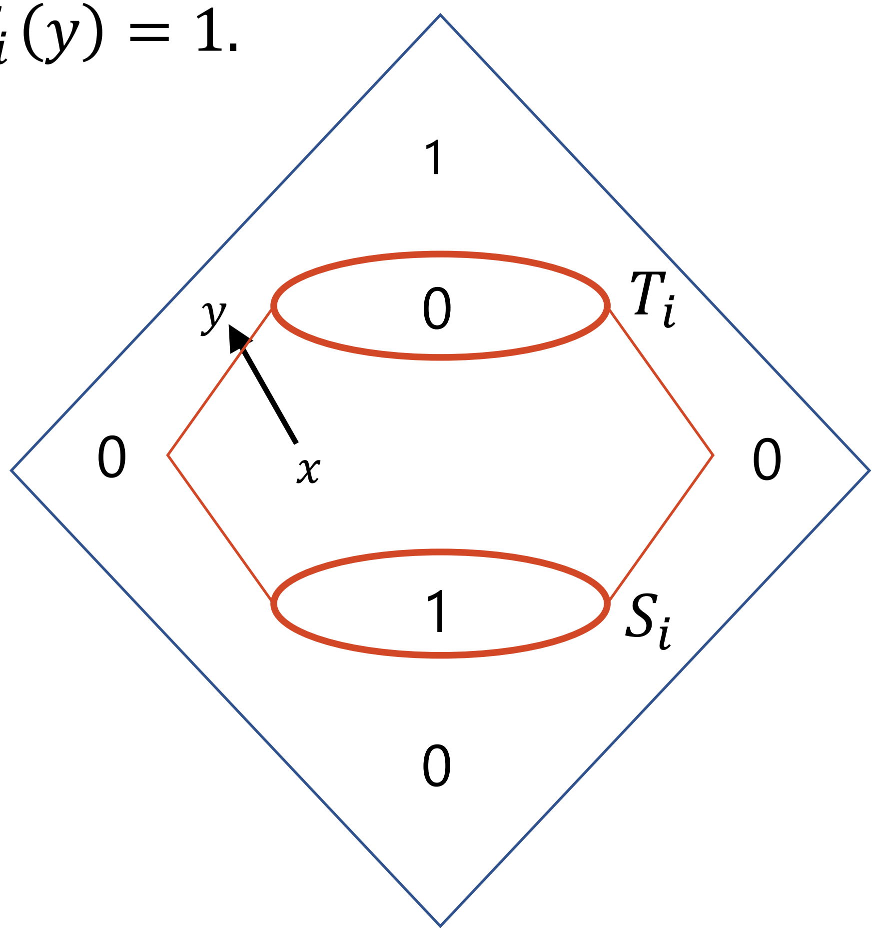
$$\sum_{i\in[k]} \mathrm{dist}(f_i, \mathrm{mono}) \geq \sum_{i\in[k]} |M_i| \geq |M| \geq \frac{1}{2}\, \mathrm{dist}(f, \mathrm{mono})$$

Step 4: Proof of BD Theorem

(2) Edges violated by f_i are contained in $\mathrm{Sweep}(S_i, T_i)$

Consider edge $x \to y$ not in $\mathrm{Sweep}(S_i, T_i)$

y above $\mathrm{Sweep}(S_i, T_i)$, $f_i(y) = 1$.

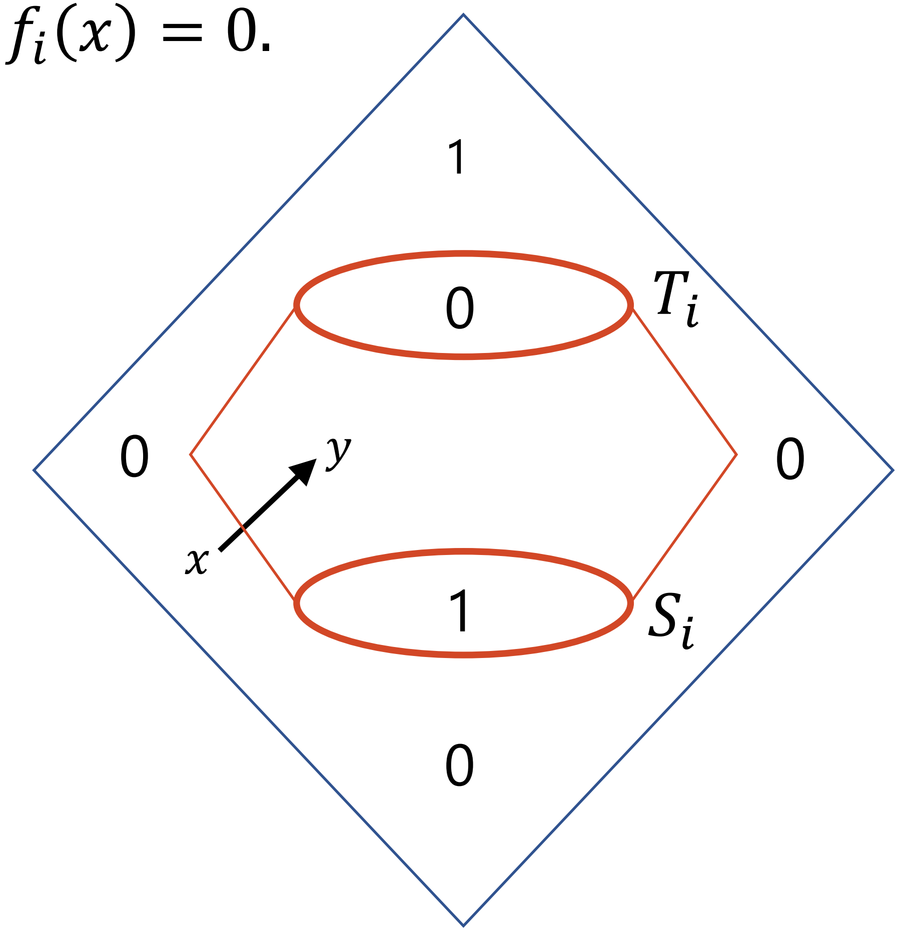


Step 4: Proof of BD Theorem

(2) Edges violated by $\boldsymbol{f_i}$ are contained in $\mathbf{Sweep}(\boldsymbol{S_i}, \boldsymbol{T_i})$

Consider edge $x \to y$ not in $\mathrm{Sweep}(S_i, T_i)$

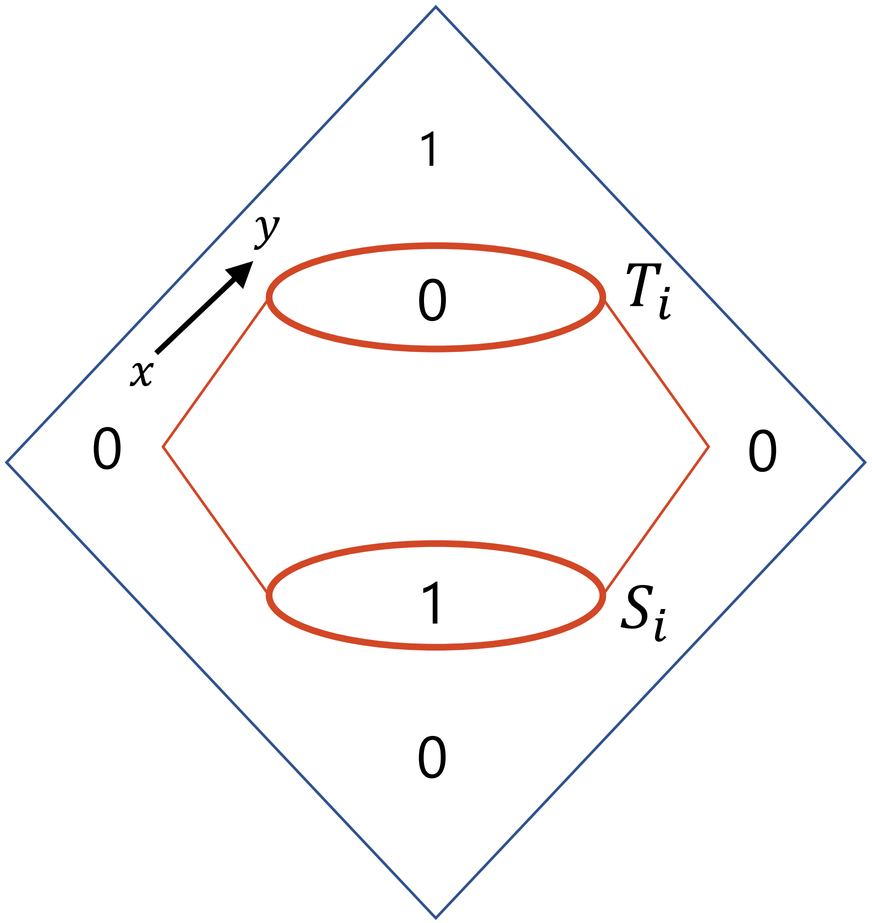
x below $\mathrm{Sweep}(S_i, T_i)$, $f_i(x) = 0$.



Step 4: Proof of BD Theorem

(2) Edges violated by f_i are contained in $\mathrm{Sweep}(S_i, T_i)$

Consider edge $x \to y$ not in $\mathrm{Sweep}(S_i, T_i)$



Step 4: Proof of BD Theorem

(3) Edges violated by f_i are violated by f

Consider edge $x \to y$ violated by f_i (in $\mathrm{Sweep}(S_i, T_i)$)

$f_i(x) = 1$, $f_i(y) = 0$

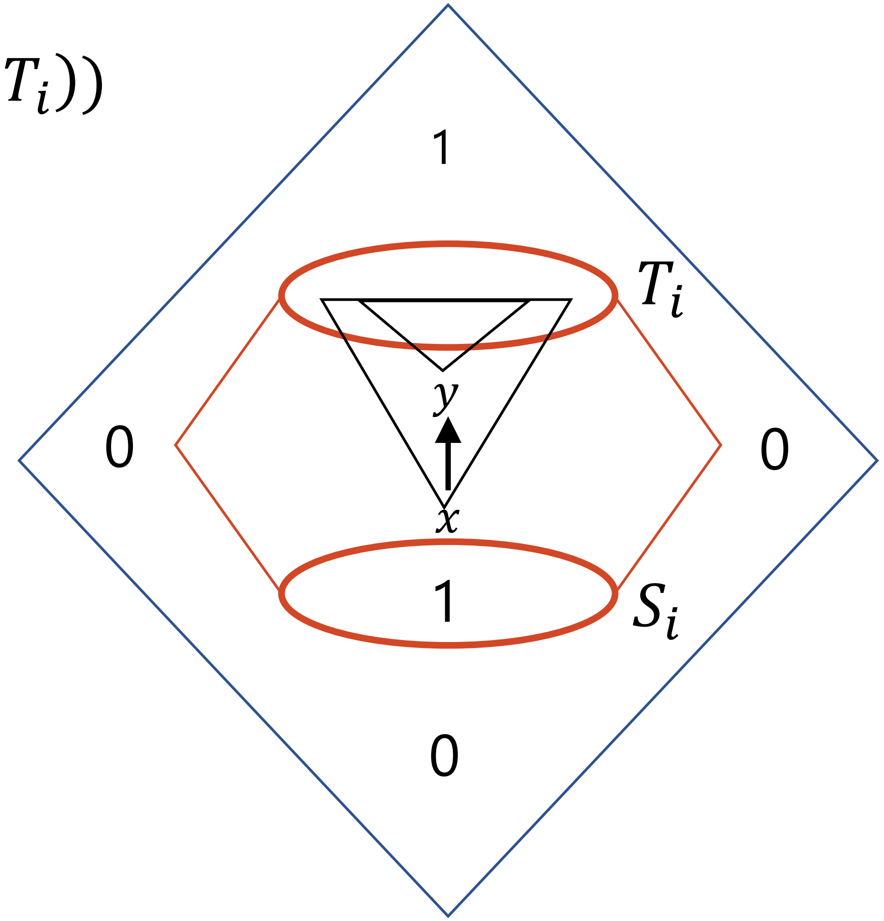
For $t \in T_i$ such that $y \preccurlyeq t$, then $x \preccurlyeq t$.

Therefore:

threshold for x threshold for y

$$f(x) > \max_{t \in T_i, x \preccurlyeq t} f(t) \geq \max_{t \in T_i, y \preccurlyeq t} f(t) \geq f(y)$$

$\Rightarrow$ Edge $x \to y$ violated by f



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- ✓ Prove desired properties of f_i.

$\Rightarrow$ Main inequality $\sum_{x \in \{0,1\}^d} \left[\sqrt{I_f^-(x)}\right] = \Omega\big(\mathrm{dist}(f, \mathrm{mono})\big)$

Conclusion

- Improved sublinear algorithms for monotonicity.
- Generalized isoperimetric inequalities.
- Proved the Boolean Decomposition Theorem.

Open Question. Do the isoperimetric inequalities hold for other domains?

- Specifically, the hypergrid domain $[n]^d$.
- Margulis type inequality holds [Black Chakrabarty Seshadhri '18] . What about Talagrand?
- It would suffice to show such inequality for the Boolean case.
- Use our BD Theorem to generalize to real-valued functions.
- Improve algorithms for monotonicity testing on hypergrid.