Sublinear-time Computation in the Presence of Online Erasures

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Overview

Goal of this work: study basic computational tasks in extremely adversarial environments

- property testing tasks
- algorithm has query access to a very large dataset via an oracle
- answer yes/no questions about global properties of the dataset

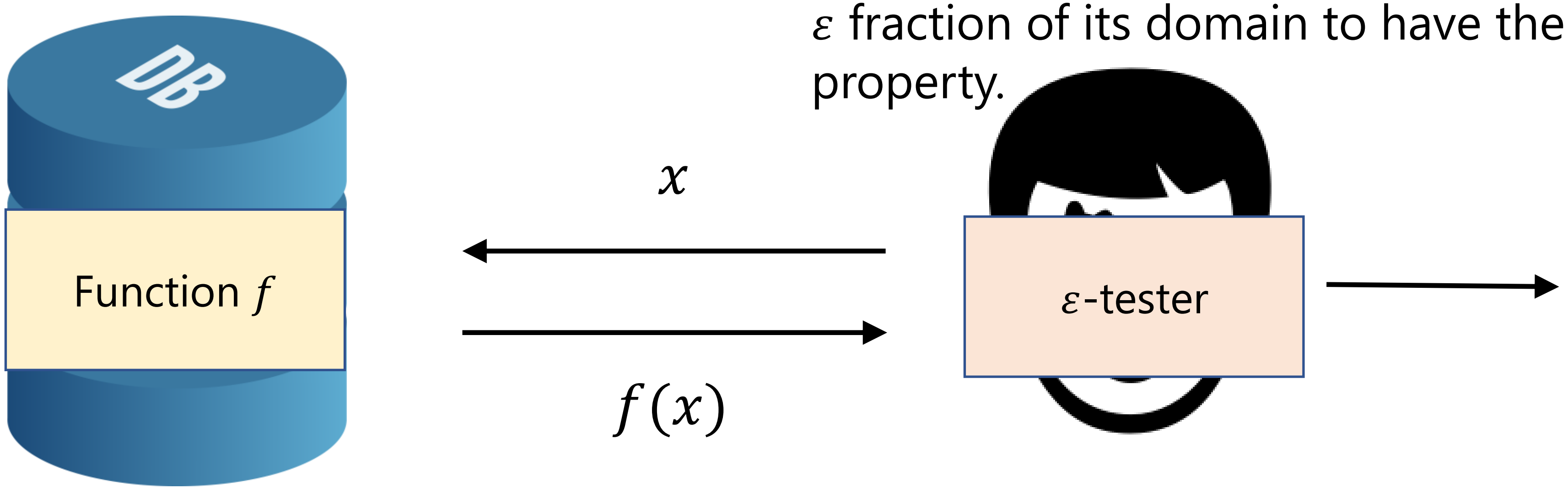
- an adversary/oracle makes changes to the dataset
- we focus on erasures
- the changes happen "online", as the dataset is being queried
- adversary can adapt to actions of algorithm

Standard property testing model

[Rubinfeld Sudan '96] [Goldreich Goldwasser Ron '98]

Does f have a property,
or is it ε-far from having the property?

ε-far: f must be modified in at least ε fraction of its domain to have the property.



- Accept if f has property
- Reject w.h.p. if f is ε-far from property

- Interested in query complexity of tester
- #queries should be sublinear in size of domain of f

Standard property testing model

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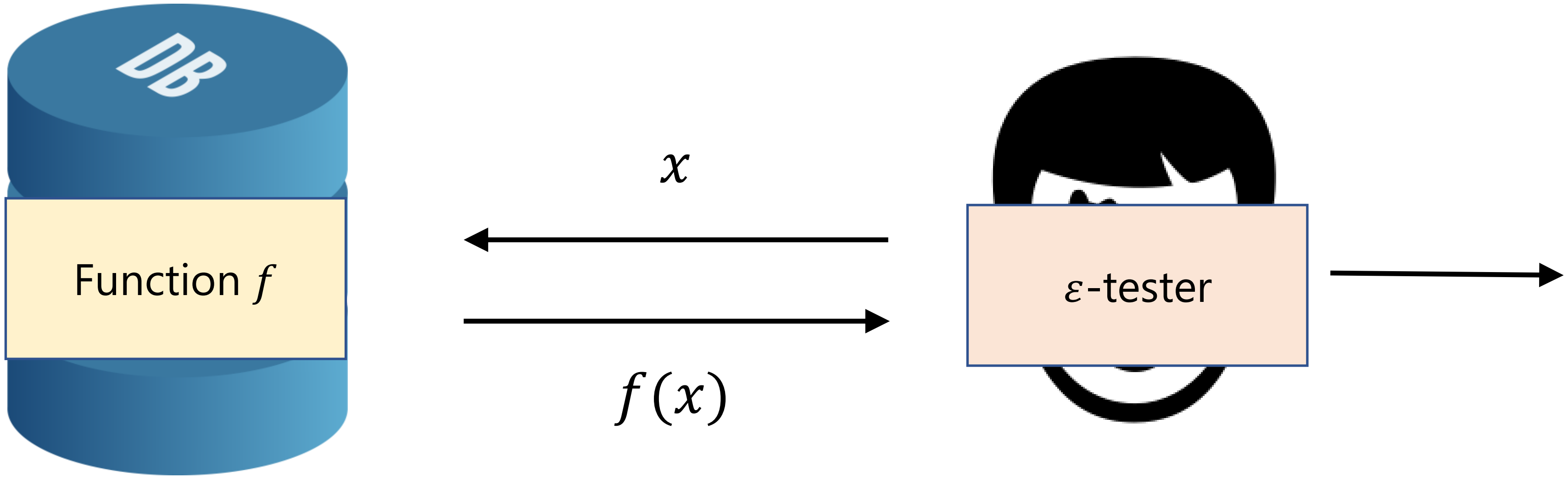

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We want to make tester robust to:

- data is missing/ corrupted
- data is erased/ corrupted adversarially
- privacy concerns

[Parnas Ron Rubinfeld '06]

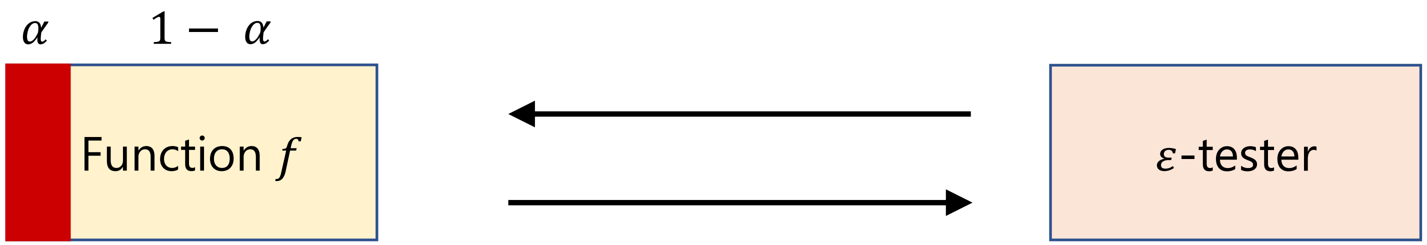
Tolerant property testing

[Dixit Raskhodnikova Thakurta Varma '18]

Erasure-resilient property testing

Offline Erasures Model

- Property testing with erasures was first studied by Dixit Raskhodnikova Thakurta Varma '18
- Oracle erases at most α fraction of the input values, before algorithm makes any queries.
- What if erasures happen during the querying process?

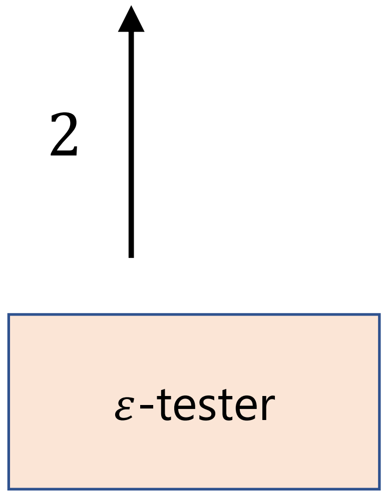


Online Erasures Model

Oracle can erase t entries after answering each query of the tester

$t = 1$

x	1	2	3	4	5	6	7	8
$f(x)$	$f(1)$	$f(2)$	.	.	.	.	.	$f(8)$



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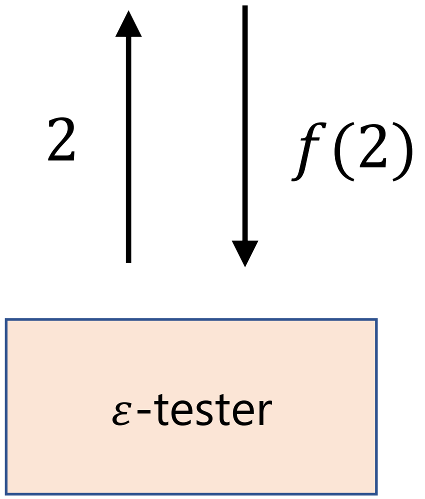

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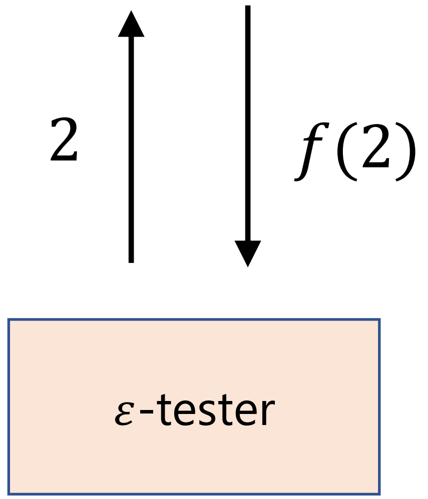

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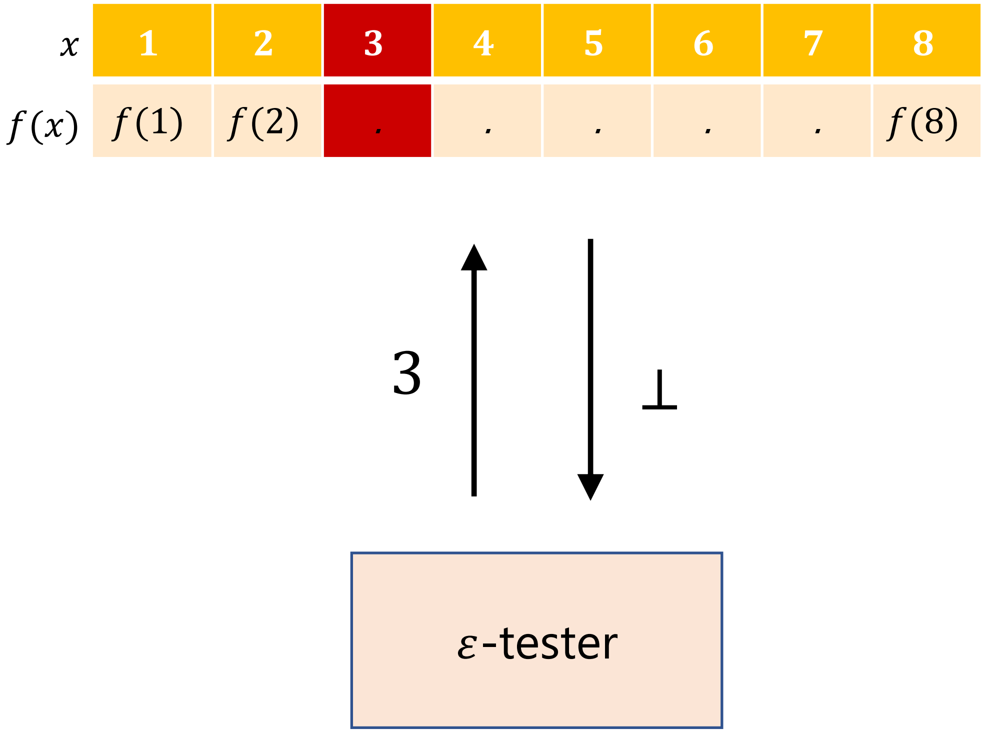

Does f have a property,
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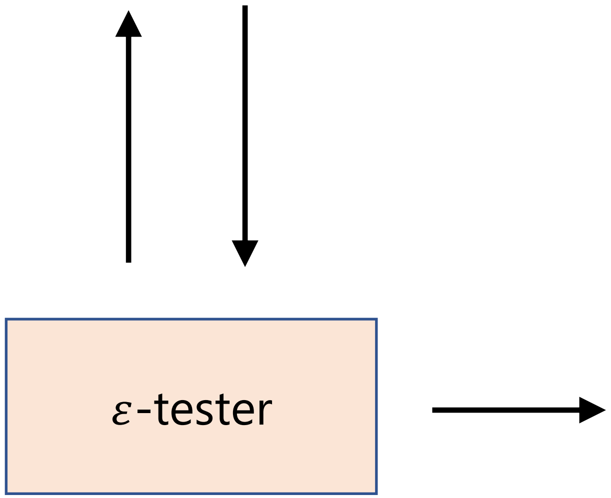
Online-Erasure-Resilient Tester

Oracle can erase t entries after answering each query of the tester

x	1	2	3	4	5	6	7	8
$f(x)$	$f(1)$	$f(2)$	.	.	.	.	.	$f(8)$



- Accept if f has property.
- Reject whp if f is ε-far from property.

Online Erasures Model

Assumptions:

- Oracle knows the description of the algorithm
- Oracle does not have access to random coins of algorithm

Example:

"Query location 1 with probability ½ and query location 2 with remaining probability"

Questions about the model?

Oracle can erase t entries after answering each query of the tester

x	1	2	3	4	5	6	7	8
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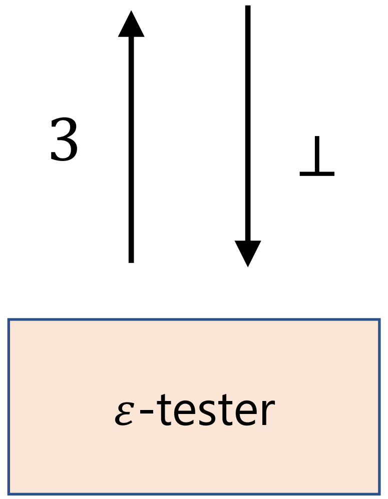

Does f have a property,
or is it far from having the property?

Motivating Scenarios

- Individuals request that their data be removed from a dataset
  - They are prompted to restrict access to their data after noticing an inquiry into their or other's data (online)
  - Adversarial assumption allows us to study worst-case
- In a criminal investigation / fraud detection setting, adversary reacts by erasing data after some of their records are pulled by authorities
- In legal setting, adversary is served a subpoena; after answering the query, they can destroy related evidence not involved in the subpoena
  - In our model, adversary can make erasures only after answering the query of the algorithm

Results

- Some properties can be tested with the same query complexity as in the standard model:
  - linearity and quadraticity (for constant erasure budget t)
- For linearity, we show matching upper and lower bounds in terms of t
- Some properties are impossible to test, even for $t = 1$: sortedness of arrays
  - The structure of violations to the property plays a role in determining testability

Plan

- Show the tester for linearity (with a light proof)
- Show the lower bound for linearity
- Show idea behind tester for quadraticity
- Show the impossibility of testing sortedness

Linearity

Function $f: \{0,1\}^d \to \{0,1\}$ is linear if can be expressed as sum of $x[i]$, $i \in [d]$

Equivalently, if $f(x) + f(y) = f(x+y)$ for all x, y in domain.

Standard Model	**Online-Erasures Model**
[Blum Luby Rubinfeld '93] [Bellare Coppersmith Hastad Kiwi Sudan '96] $O\left(\frac{1}{\varepsilon}\right)$ queries	This work $\tilde{O}\left(\frac{\log t}{\varepsilon}\right)$ queries

BLR Tester:

- Sample $x, y \sim \{0,1\}^d$.
- Query $f(x), f(y), f(x+y)$.
- Reject if $f(x) + f(y) \neq f(x+y)$.

Issue with standard linearity tester:

- Query x. Receive $f(x)$.
- Query y. Receive $f(y)$.
- Oracle erases $x + y$.

If $f: \{0,1\}^d \to \{0,1\}$ is ε-far from linear then an ε-fraction of pairs (x, y) violate linearity.

Linearity

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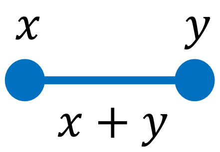
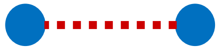


2-player game:
1) Player 1 draws a vertex or edge connecting two vertices in blue
2) Player 2 draws an edge between existing vertices in red



Can you come up with winning strategy for player 1?

Linearity

Function $f: \{0,1\}^d \to \{0,1\}$ is linear if can be expressed as sum of $x[i],\ i \in [d]$

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If $f: \{0,1\}^d \to \{0,1\}$ is ε-far from linear then an ε-fraction of pairs (x, y) violate linearity.

Thm. If $f: \{0,1\}^d \to \{0,1\}$ is ε-far from linear then, for all even k, an ε-fraction of k-tuples $(x_1, x_2, \ldots, x_k)$ violate linearity.

$f(x_1) + \cdots + f(x_k) \neq f(x_1 + \cdots + x_k)$

Proof via Fourier analysis

Linearity

Algorithm. Online-erasure-resilient linearity tester

t = erasures per query

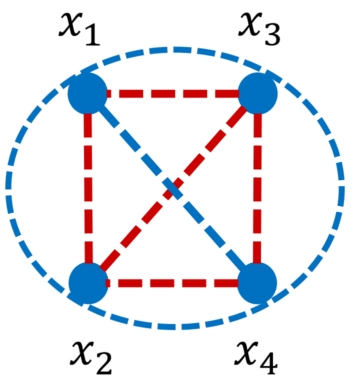
t = 2

(1) Query $q = 2\log(t/\varepsilon)$ points $x_i \sim \{0,1\}^d$

(2) Repeat $1/\varepsilon$ times:

- Sample nonempty even-sized subset I of $[q]$
- Query f at $\sum_{i \in I} x_i$
- **Reject** if $\sum_{i \in I} f(x_i) \neq f(\sum_{i \in I} x_i)$ (and all points are non-erased)

(3) **Accept**



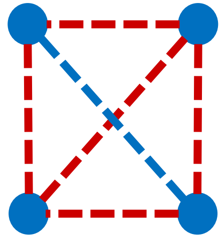
Proof. Algorithm always accepts if f is linear. Suppose f is ε-far from linear.

- Goal: obtain, nonerased, all values of some k-tuple that violates linearity.
- Step (1): All x_i are sampled iid, so they are nonerased with high probability.
- Step (2):
  - Number of even-sized subsets of $[q]$: $2^{q-1} = t^2/\varepsilon^2$
  - Expected number of violating sets (by structural Theorem): $\varepsilon \cdot 2^{q-1} = t^2/\varepsilon$
  - Number of even-sized sets spoiled by adversary: $t\left(q + \frac{1}{\varepsilon}\right) = 2t \log \frac{t}{\varepsilon} + \frac{t}{\varepsilon} \leq \frac{3t \log t}{\varepsilon}$
  - Expected fraction of nonerased violating even-sized sets $\geq \varepsilon/2$
  - After $O(1/\varepsilon)$ iteratitions, tester will sample nonerased violating sum

Linearity

Q. Why not just query sums of pairs, i.e., why do we need the structural theorem?

A. To obtain optimal dependence on t in the query complexity of the tester



Algorithm. Online-erasure-resilient linearity tester

(1) Query $q = O(t^2)$ points $x_i \sim \{0,1\}^d$

(2) Repeat $1/\varepsilon$ times:

- Sample nonempty subset I of $[q]$ of size 2.
- Query f at $\sum_{i \in I} x_i$
- Reject if $\sum_{i \in I} f(x_i) \neq f(\sum_{i \in I} x_i)$ (and all points are non-erased)

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Plan

✓ Show the tester for linearity (with a light proof)

- Show the lower bound for linearity
- Show idea behind tester for quadraticity
- Show the impossibility of testing sortedness

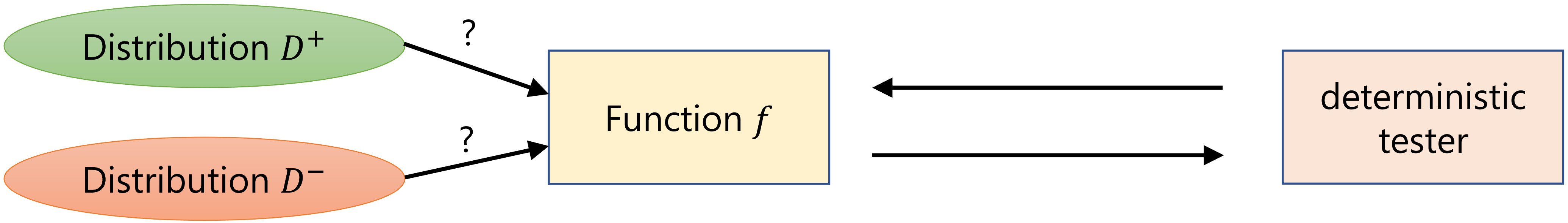
Linearity Lower Bound

Thm. Every online-erasure-resilient linearity tester must make at least $\log t$ queries.

Proof. Via Yao's minimax principle.

To show a lower bound q on randomized algorithms for testing a property it suffices to show:

- two distributions D^+ and D^- over functions f
- functions from D^+ have the property
- functions from D^- are far from the property (w.h.p.)
- a deterministic tester is given query access to f generated from D^+ or D^-
- if the tester makes $< q$ queries, it cannot decide between D^+ and D^- with low prob. of error



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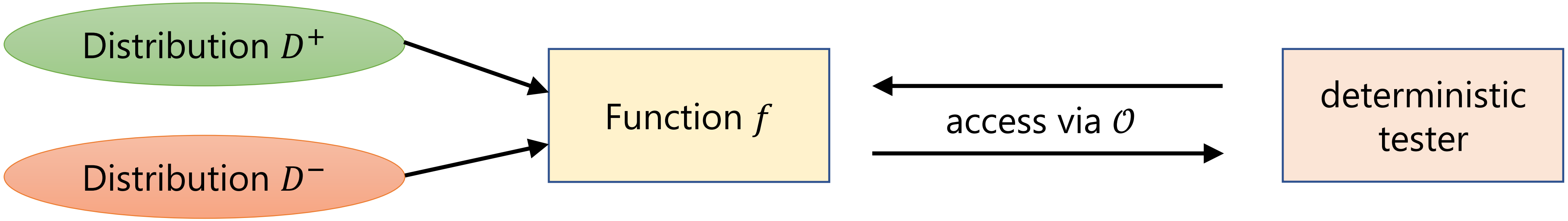
To show a lower bound q on randomized algorithms for testing a property it suffices to show:

- two distributions D^+ and D^- over functions f
- functions from D^+ have the property
- functions from D^- are far from the property (w.h.p.)
- an erasure strategy for t-online-erasure oracle $\mathcal{O}$
- a <u>deterministic</u> tester is given query access via $\mathcal{O}$ to f generated from D^+ or D^-
- if the tester makes $< q$ queries, it cannot decide between D^+ and D^- with low prob. of error



Linearity Lower Bound

Thm. Every online-erasure-resilient linearity tester must make at least $\log t$ queries.

D^+: random linear

D^-: random function

Proof. Via Yao's minimax principle.

- D^+: Uniform distribution over linear functions on $\{0,1\}^d$
- D^-: Uniform distribution over all Boolean functions on $\{0,1\}^d$ ($\frac{1}{4}$-far from linear w.h.p)
- Oracle $\mathcal{O}$: erase t sums of previous queries of the tester (in some specific order)
- If tester makes $q < \log t$ queries, with t erasures the oracle can erase $t > 2^q$ points
- i.e., oracle erases all sums of queried elements
- Tester only sees linearly independent vectors from $\{0,1\}^d$
- For a uniformly random linear function, the distribution of values over a set of linearly independent vectors is uniform
- A linear function is fully specified by its values on the basis vectors for $\{0, 1\}^d$
- If tester makes $< \log t$ queries, it cannot distinguish D^+ from D^-

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Quadraticity

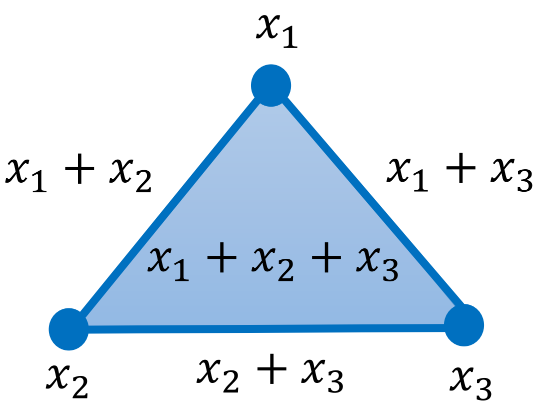
Function $f: \{0,1\}^d \to \{0,1\}$ is quadratic if can be expressed as polynomial of degree at most 2
e.g., $f(x) = x[1]x[2] + x[3]$

Standard Model	Online Erasures Model
[Alon Kaufman Krivelevich Litsyn Ron '05] [Bhattacharyya Kopparty Schoenebeck Sudan Zuckerman '10] $O\left(\frac{1}{\varepsilon}\right)$ queries	This work $O\left(\frac{1}{\varepsilon}\right)$ queries for constant t Doubly exponential in t

Tester:
- Sample $x_1, x_2, x_3 \sim \{0,1\}^d$
- For all nonempty $S \subseteq [3]$, query $\sum_{i \in S} x_i$
- Reject if the sum of f on 7 queries is 1.

Raise of hands: Can one modify this tester to work with erasures?

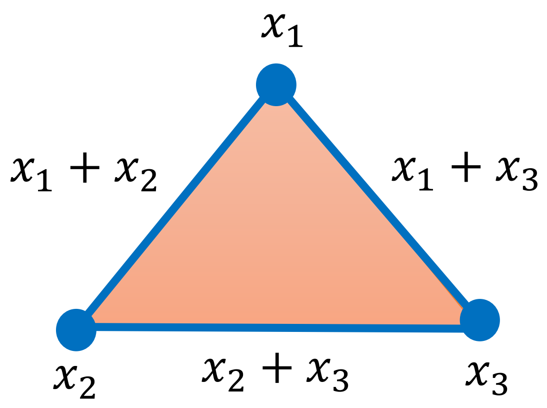
Recall 2 player game.



Quadraticity

2-player game:

- Player 1 draws a vertex or edge connecting two vertices or colors a triangle in blue
- Player 2 draws an edge between existing vertices or colors a triangle in red



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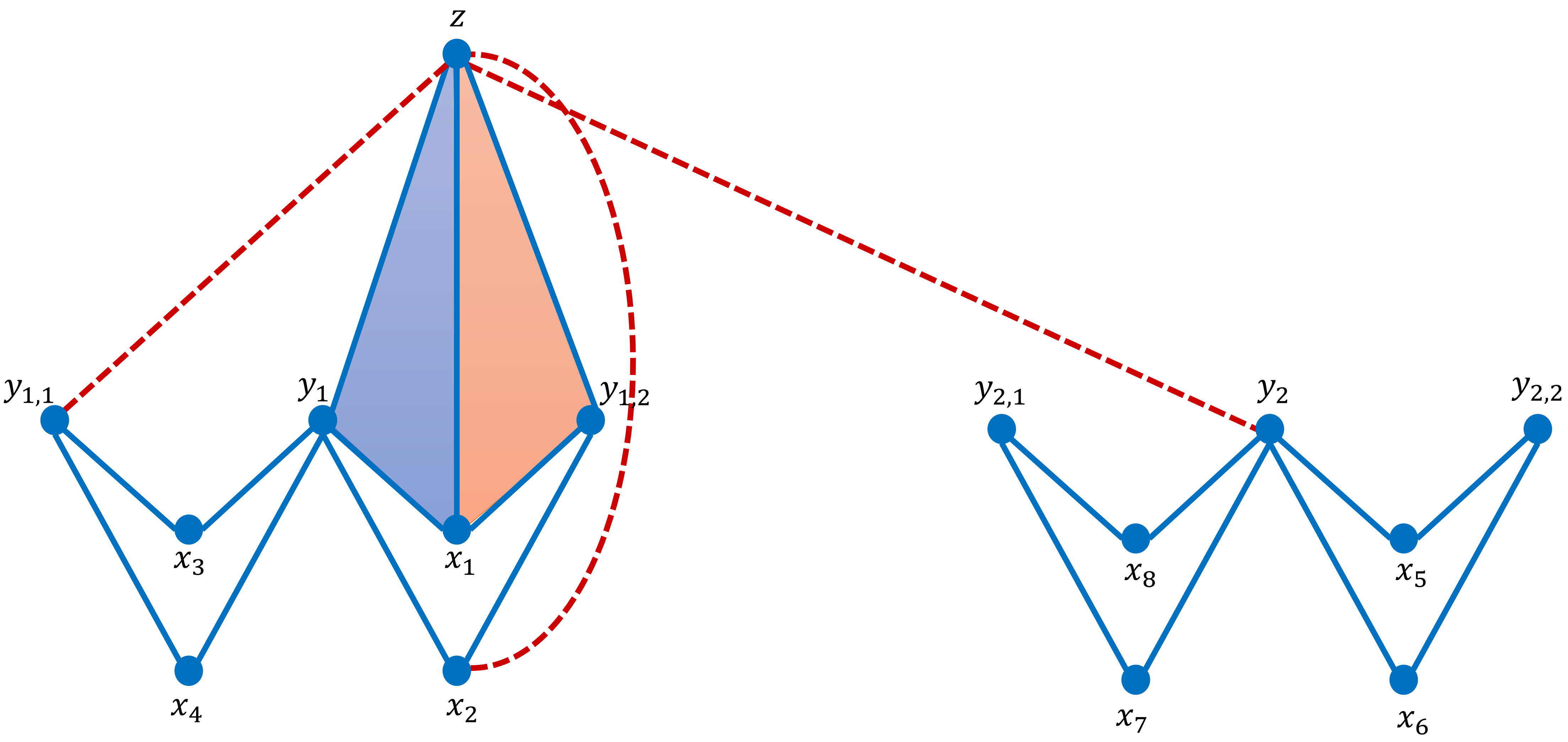

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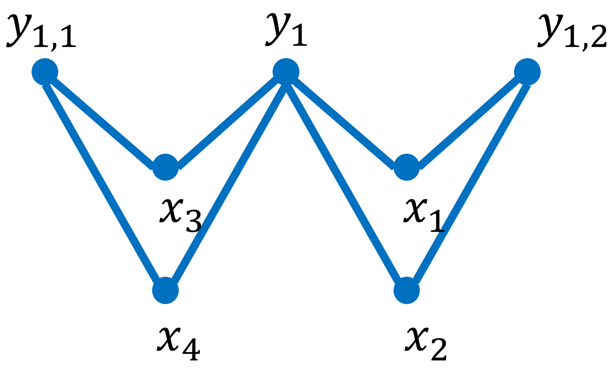

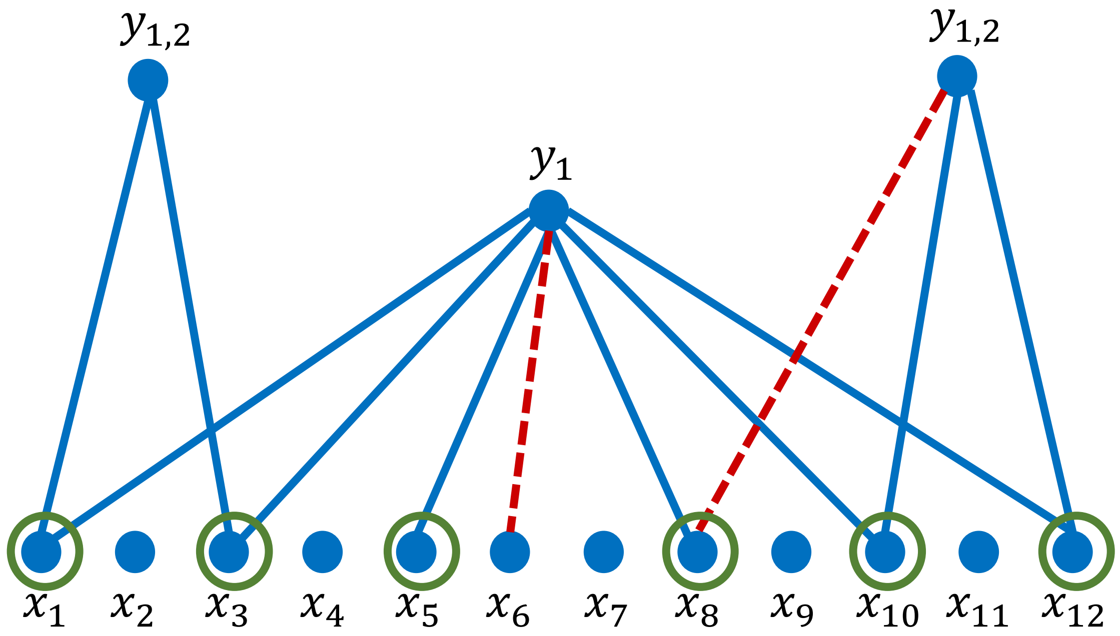

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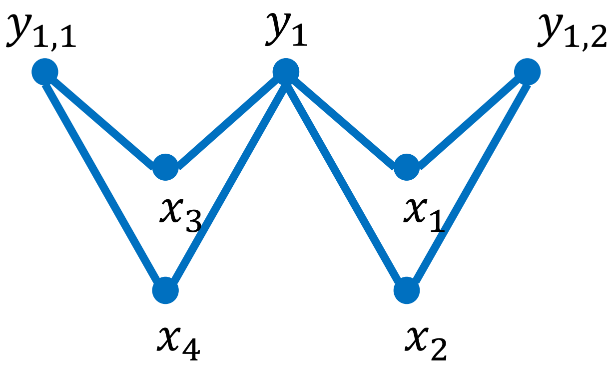
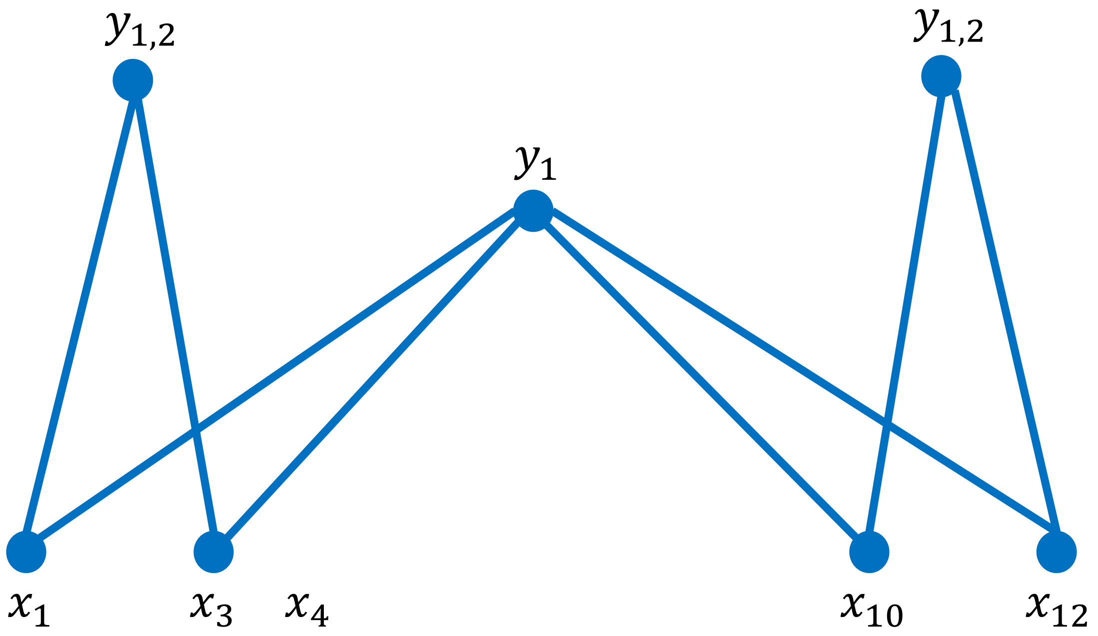


From the game to the algorithm:
- Probability that the queries made by the the tester are nonerased when queried?
- Probability that the "triangle" completed violates quadraticity?

Generalize to t: A strategy for Player 1 with $t^{O(t)}$ moves



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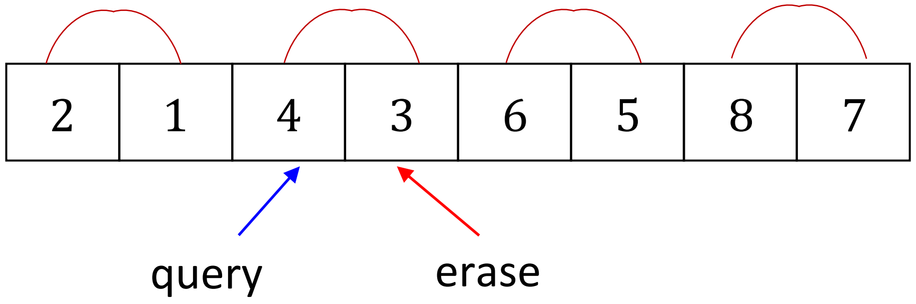
Sortedness

Array $f: [n] \to \mathbb{N}$ is sorted if $f(x) \leq f(y)$ for all $x \leq y$

Standard Model	Offline-Erasures Model	Online-Erasures Model
[Ergun Kannan Kumar Rubinfeld Viswanathan '00] [Fischer Lehman Newman, Raskhodnikova Rubinfeld Alex Samorodnitsky '04][Fischer '06] [Bhattacharyya Grigorescu Jung Raskhodnikova Woodruff '12] [Chakrabarty Seshadhri '18][Belovs '18] $\Theta(\log \varepsilon n / \varepsilon)$ queries $O(\sqrt{n/\varepsilon})$ uniform iid queries	[Dixit Raskhodnikova Thakurta Varma '18] $O(\log n / \varepsilon)$ queries	This work Impossible to test



- array is ½-far from sorted
- all violations are disjoint
- in linearity and quadraticity, violations overlap with each other

Plan

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- ✓ Show the lower bound for linearity
- ✓ Show idea behind tester for quadraticity
- ✓ Show the impossibility of testing sortedness

Conclusions & Open Questions

- Designed efficient testers for several important properties (linearity and quadraticity)
- Showed tight bounds for testing linearity in terms of erasure budget t
- Showed that some basic properties cannot be tested in our model, even for $t = 1$.

- Sortedness can be tested in the offline erasures model, but not in the online erasures model.
  - Is there a property that has smaller query complexity in online model vs offline model?
- Is there a tester for testing that a function is polynomial of degree at most k for $k \geq 3$?
  - In standard model this is possible with $O(2^k/\varepsilon)$ queries
- What is the query complexity for testing quadraticity in terms of t?
  - Current tester has doubly exponential dependence on t