CS 237: Probability in Computing
Instructors: Byers and Januario

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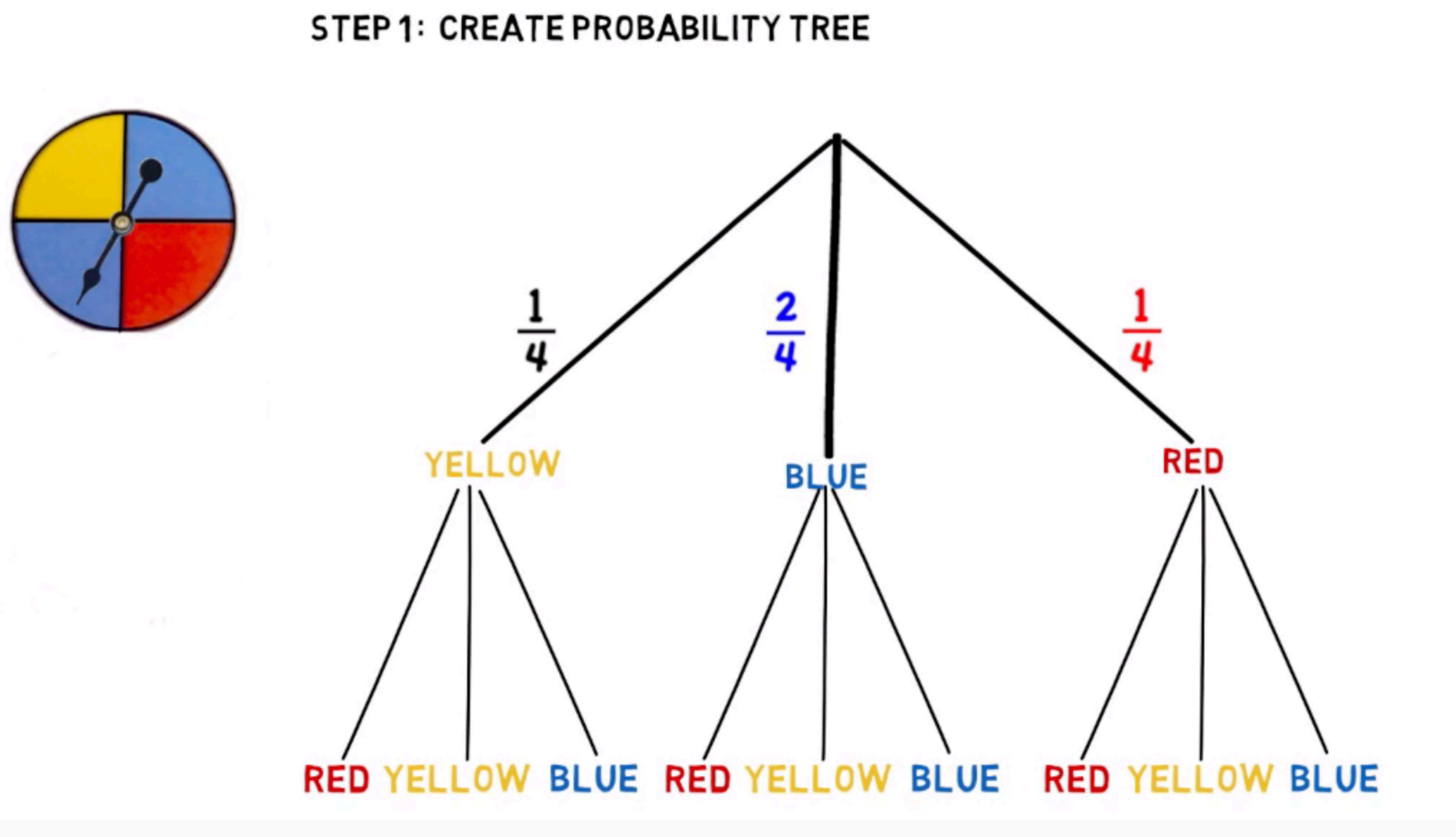


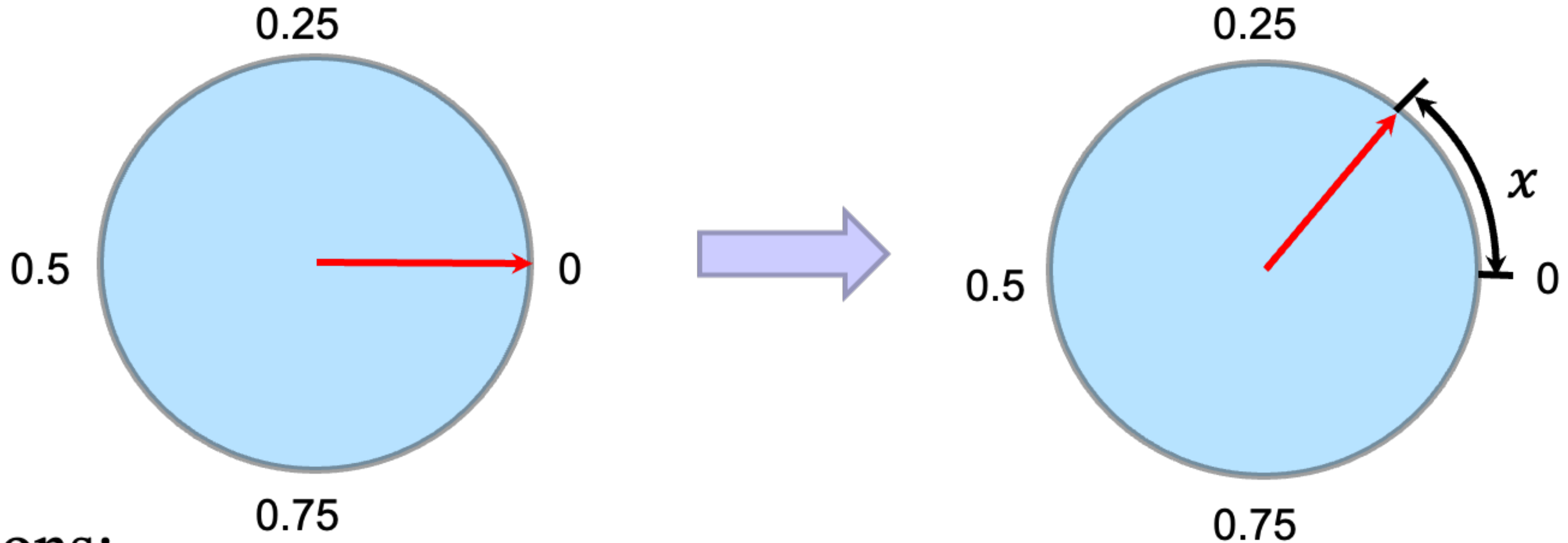
From https://www.youtube.com/watch?v=UxiCV8RroKU

SAMPLE SPACES

- Previously:
  - Discrete sample spaces: outcomes are listable (toss a coin, roll two dice, etc.)
    - We assign a probability to each outcome
    - The probability of an event is the sum of the probabilities of the outcomes comprising the event
- Today:
  - Continuous sample spaces: outcomes are not listable
    - Examples: bank account values, average test scores, submission times
    - Infinite number of possible outcomes
    - How do we work with probability functions now?

CONTINUOUS PROBABILITY SPACES

The Spinner Experiment

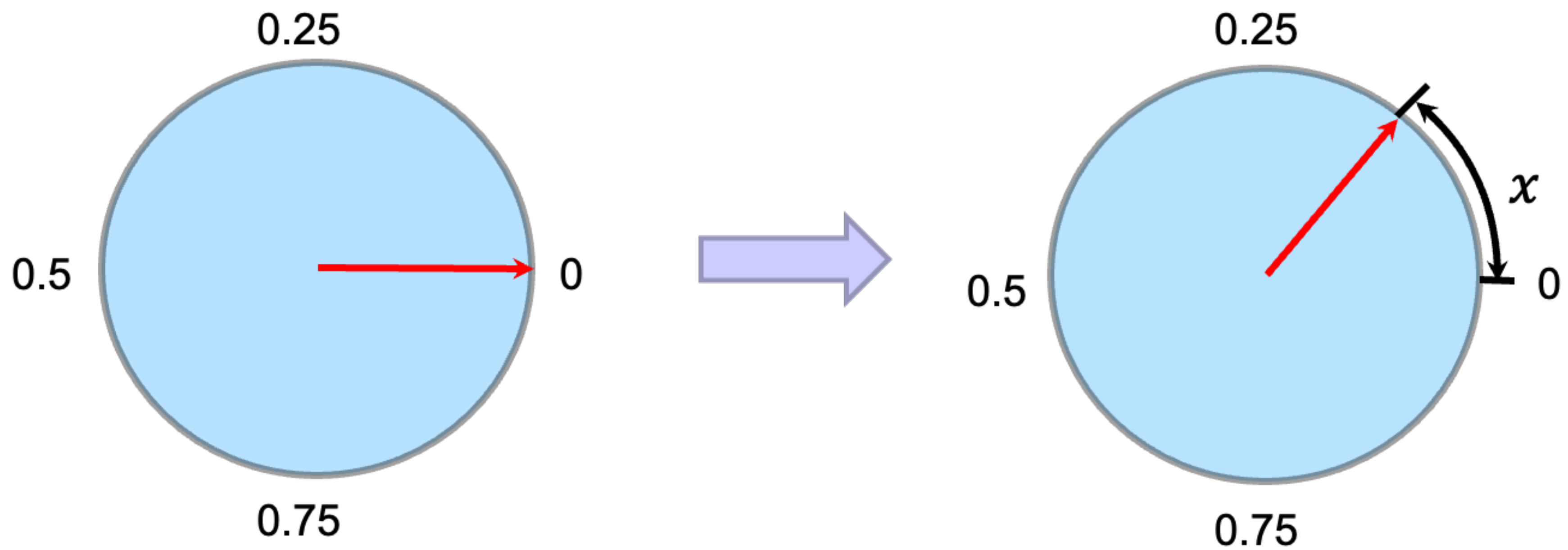


Assumptions:

- The circumference of the circle is 1
- When you spin the pointer, it is equally likely to end up anywhere around the circle

CONTINUOUS PROBABILITY SPACES

The Spinner Experiment



- Experiment: spin the pointer, see where it lands!
- Outcome: real number $x \in [0,1]$
- Sample space: $\Omega = [0,1]$

ASK THE AUDIENCE

- What is the probability that the dial lands in [0, 1/8]?

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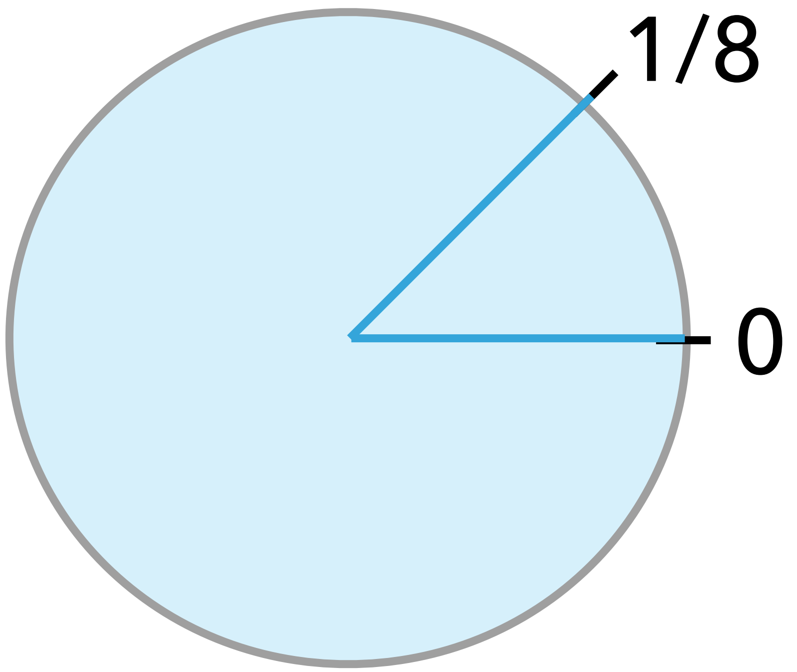
(A) 1/8	(B) 0
(C) $\pi/8$	(D) none of the above



The Spinner Experiment

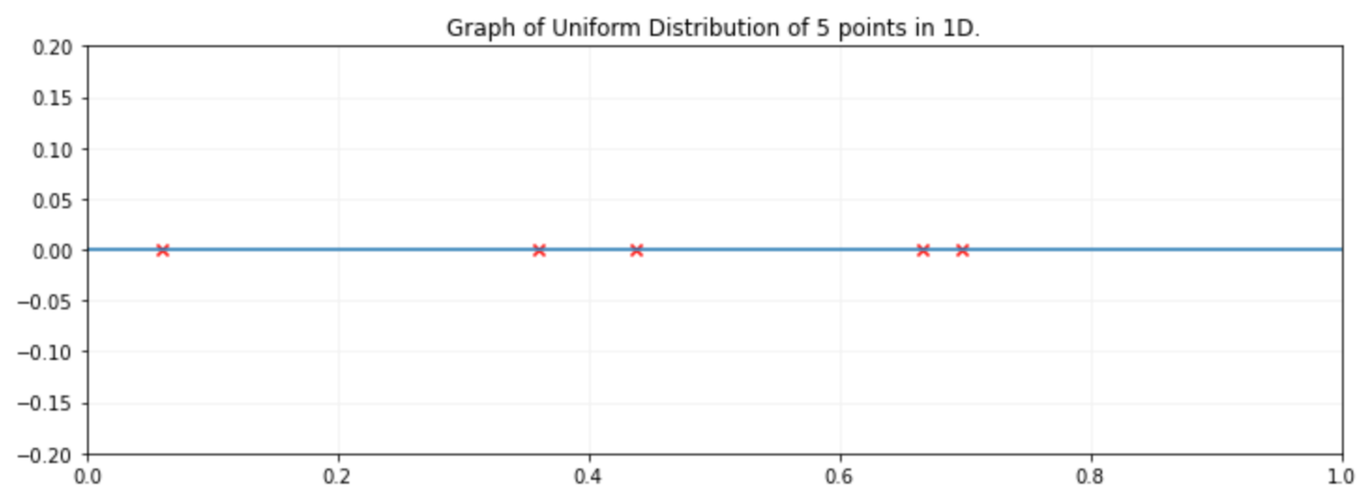
This is implemented by
the numpy function random()

```
from numpy.random import random

for k in range(5):
    print(random())
```

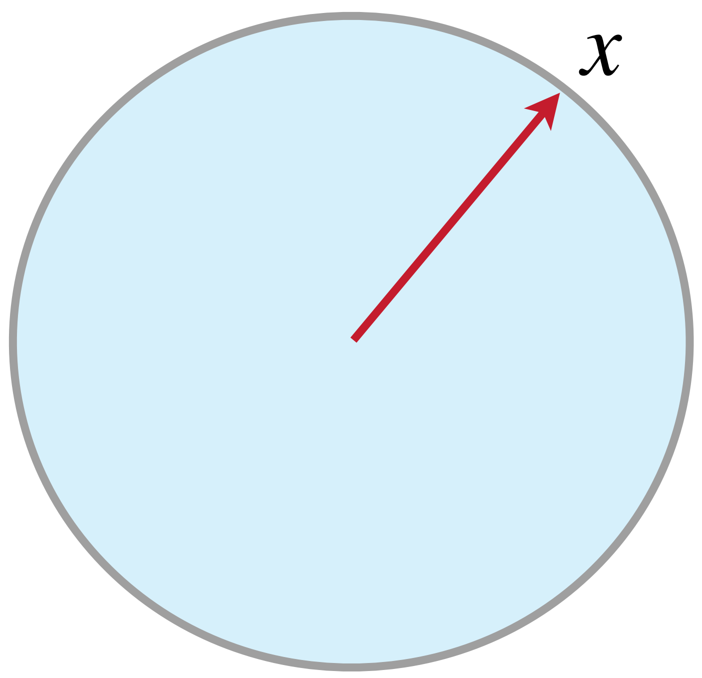
```
0.2880928755517823
0.7162125723051093
0.6655015245054124
0.554162601188949
0.28532267939275413
```

```
[0.359507900573786, 0.43703195379934145, 0.6976311959272649, 0.06022547162926983, 0.6667667154456677]
```

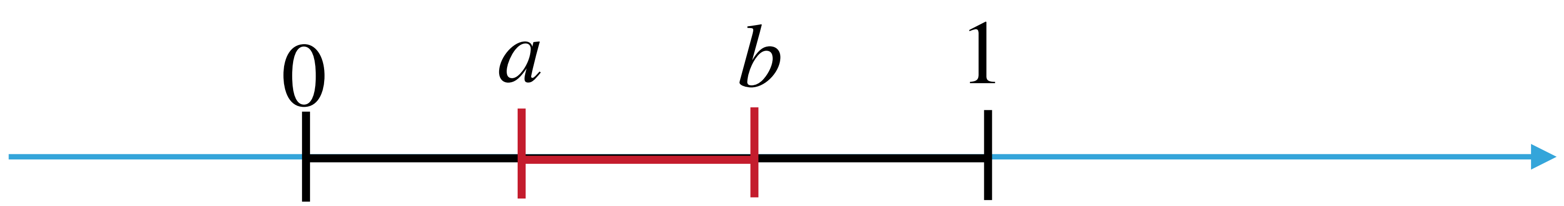


SPINNERS

- Experiment: spin the dial
- Outcome: real number $x \in [0,1]$
- Sample space: $\Omega = [0,1]$
- Probability function: for all $[a, b] \subseteq [0,1]$



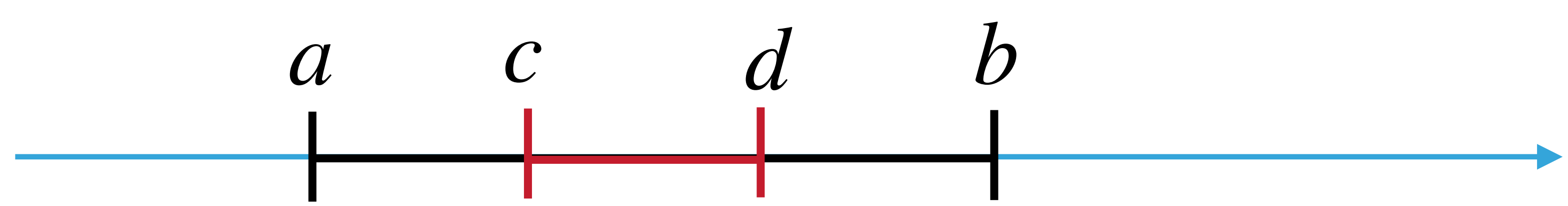
$$\Pr(x \in [a, b]) = \frac{\text{length of } [a, b]}{\text{length of } [0,1]} = b - a$$



UNIFORM SAMPLING FROM AN INTERVAL

- Experiment: sample a point uniformly from $[a, b]$
- Outcome: real number $x \in [a, b]$
- Sample space: $\Omega = [a, b]$
- Probability function: for all $[c, d] \subseteq [a, b]$, we have

$$\Pr(x \in [c, d]) = \frac{\text{length of } [c, d]}{\text{length of } [a, b]} = \frac{d - c}{b - a}$$



UNIFORM SAMPLING FROM AN INTERVAL

- Experiment: choose uniform $x \in [0,10]$


```
from numpy.random import uniform

for k in range(5):
    print(uniform(0,10))
```

```
9.458670834431103
7.821264914591
4.367437002029626
1.1280161736707706
0.4086456784560122
```

UNIFORM SAMPLING FROM AN INTERVAL

- **Experiment:** choose uniform $x \in [0,10]$

$$\Pr(x \in [0,6]) =$$

$$\Pr(x \in [0,5)) =$$

$$\Pr(x \in [0,6] \cup [8,10]) =$$

$$\Pr(x \in [0,6] \cup [5,7]) =$$

$$\Pr(x \in [-10,10]) =$$

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UNIFORM SAMPLING FROM RECTANGLES

```
# uniform sampling from unit square [0, 1]x[0, 1]
def uniform_unit_square():
    x = uniform(0, 1)
    y = uniform(0, 1)
    return (x, y)

# uniform sampling from the rectangle [a, b]x[c, d]
def uniform_rectangle(a, b, c, d):
    x = uniform(a, b)
    y = uniform(c, d)
    return (x, y)
```

DARTS WITH A SQUARE BOARD

You throw a dart at a square target 1 meter on a side.

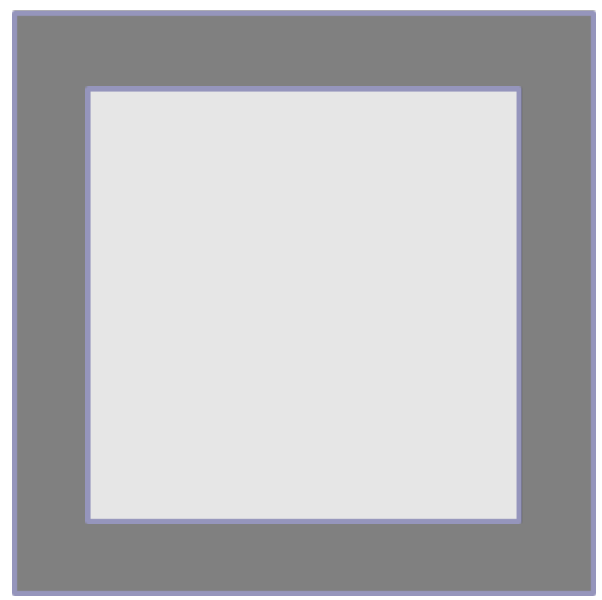
What is the probability that it lands within 0.1 m of an edge?

A. $0.8^2 = 0.64$

B. $0.9^2 = 0.81$

C. $1 - 0.8^2 = 0.36$

D. $1 - 0.9^2 = 0.19$



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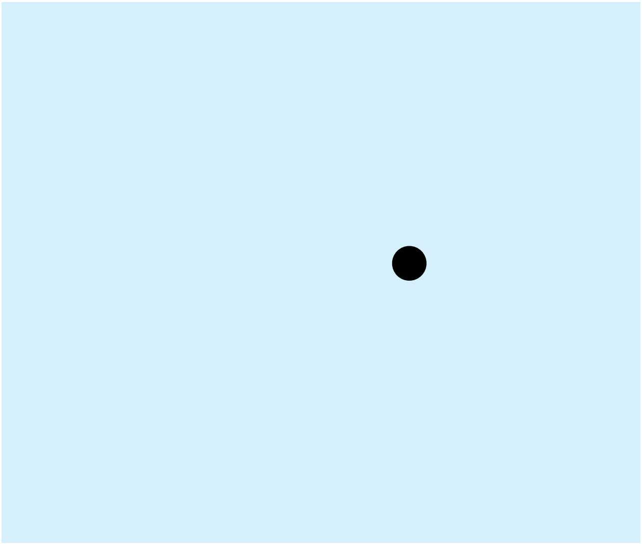
DARTS WITH A SQUARE BOARD

- Experiment: throw a dart at a square board
- Outcome: hitting point
- Sample space: the square
- Probability function:



$$\Pr(p \in P) = \frac{\text{area of } P}{\text{area of square}}$$

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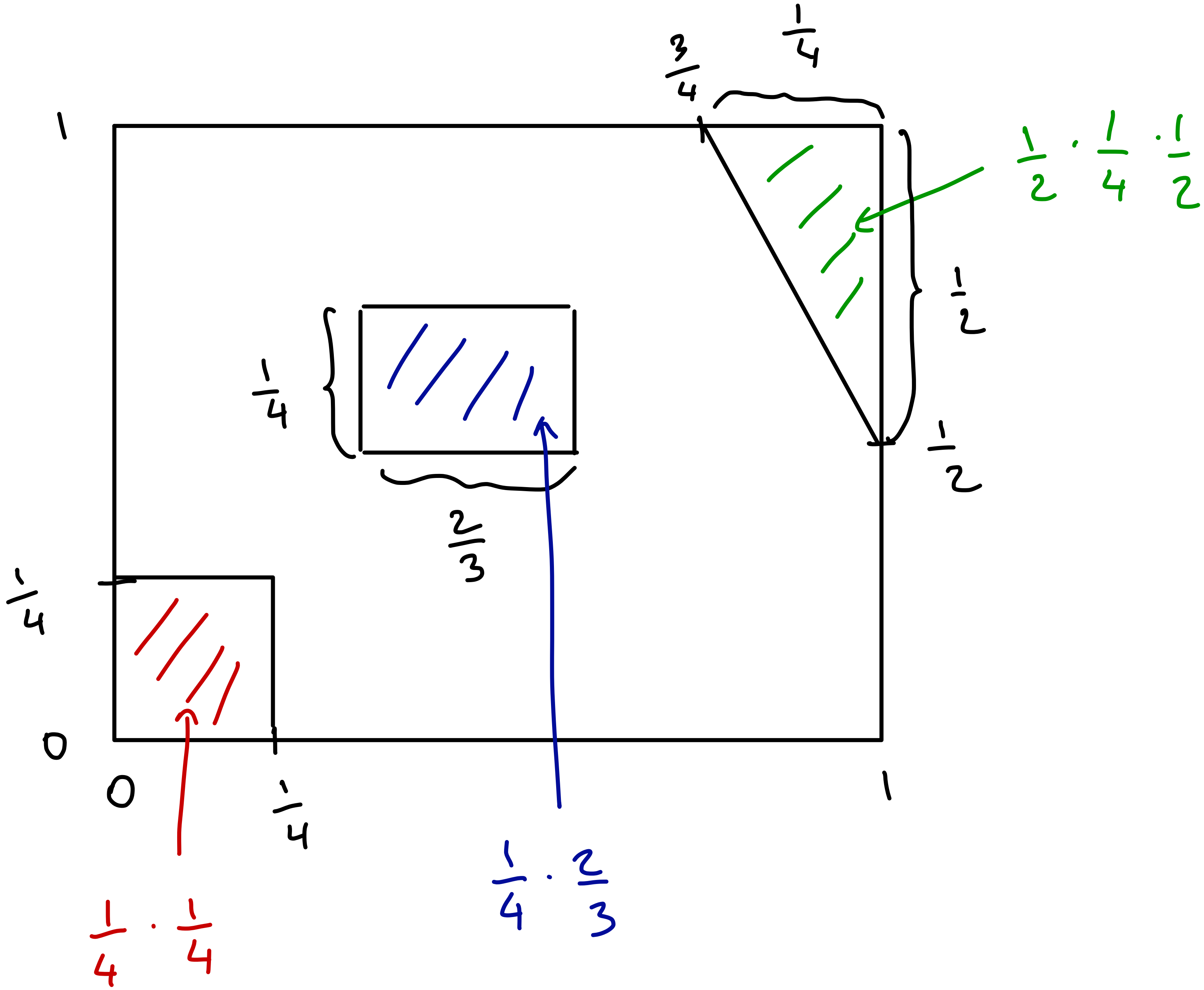
UNIFORM SAMPLING FROM THE UNIT SQUARE



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UNIFORM SAMPLING FROM RECTANGLES

```
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# uniform sampling from the rectangle [a, b]x[c, d]
def uniform_rectangle(a, b, c, d):
    x = uniform(a, b)
    y = uniform(c, d)
    return (x, y)
```

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RANDOMLY TARDY

- Tiago and John decide to work on a CS237 homework
- Each of them arrives at a time chosen uniformly at random between noon and 1pm
- Once one of them arrives, they will wait for the other for 15 minutes and leave if they do not show up
- What is the probability that Tiago and John meet up?

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A) 1/2 B) < 1/2 C) > 1/2

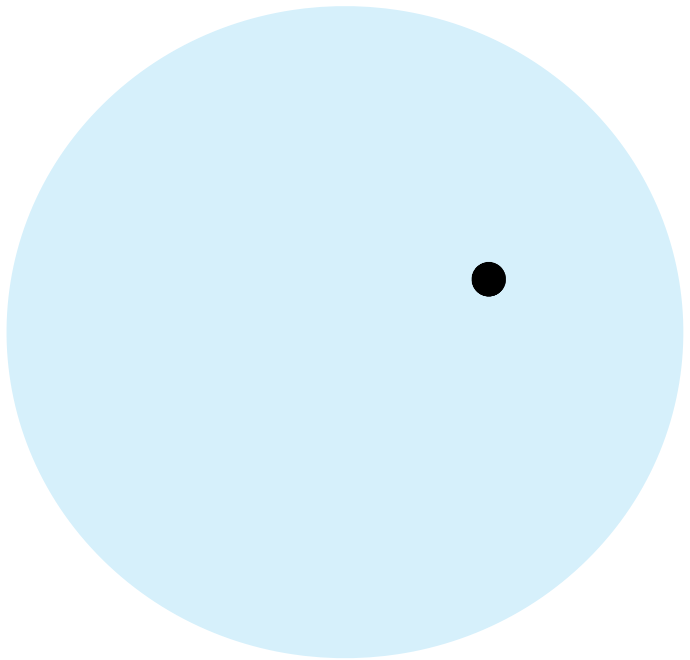
RANDOMLY TARDY

(Math on the board)

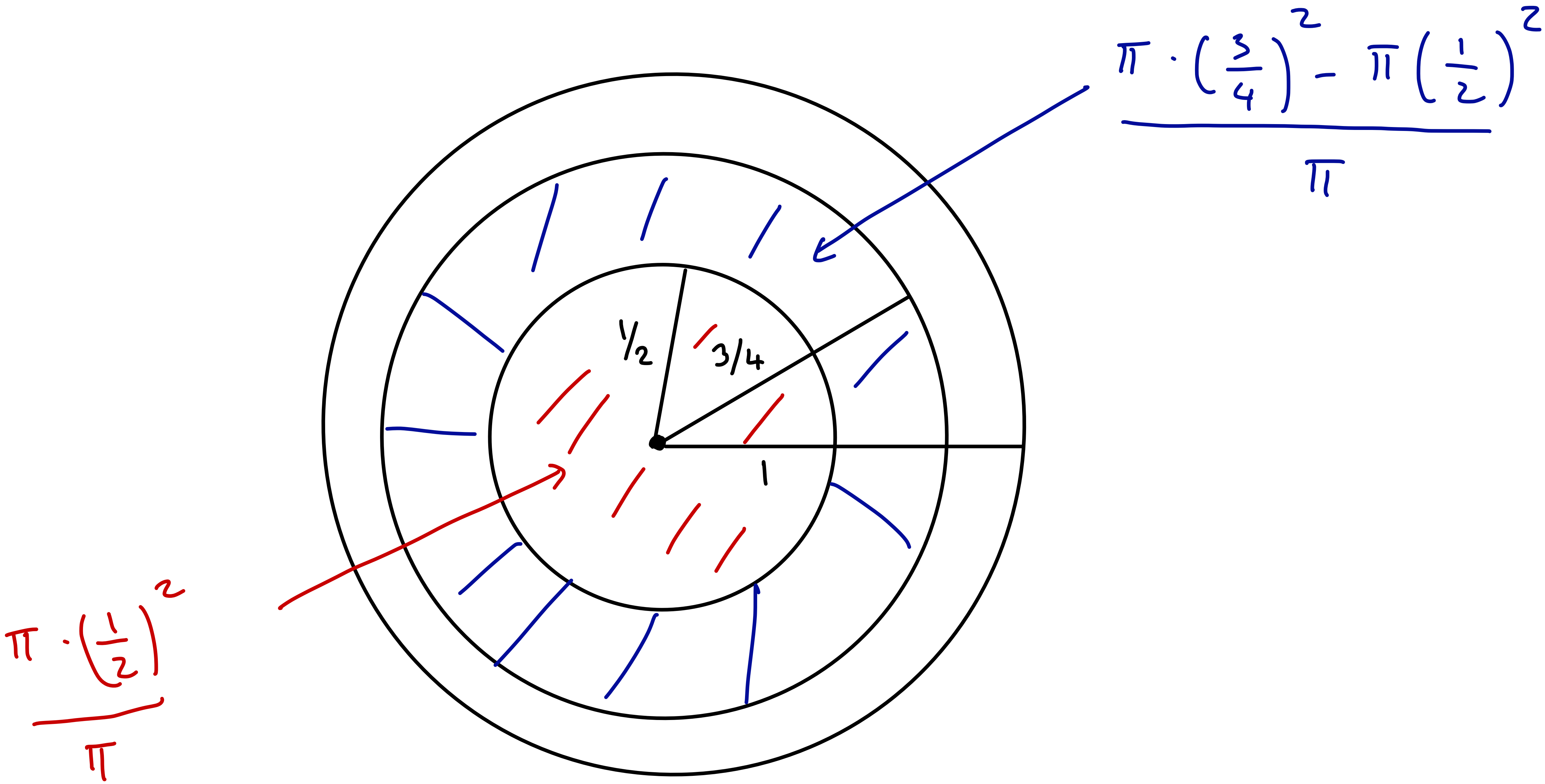
DARTS WITH A CIRCULAR BOARD

- Experiment: throw a dart at a circular board
- Outcome: hitting point p
- Sample space: the disk
- Probability function:

$$\Pr(p \in P) = \frac{\text{area of } P}{\text{area of disk}}$$



UNIFORM SAMPLING FROM THE UNIT DISK

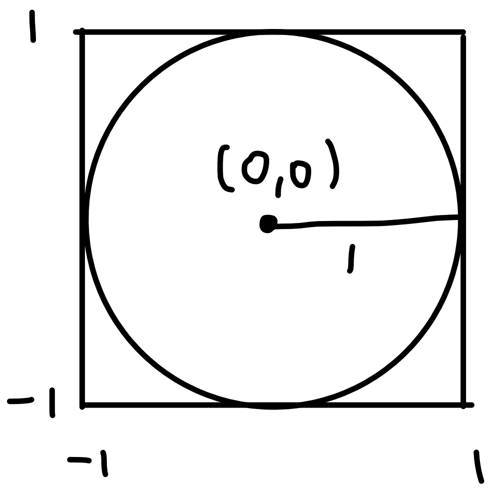


UNIFORM SAMPLING FROM THE UNIT DISK

Sample from the square enclosing the disk

If outside the disk, reject the point and repeat

Otherwise, return the sampled point




```python
# uniform sampling from a unit disk centered at (0, 0)
def uniform_unit_disk():
    while True:
        # sample from the square
        (x, y) = uniform_rectangle(-1, 1, -1, 1)
        # check if (x, y) is on the disk
        if x**2 + y**2 <= 1:
            return (x, y)
```

UNIFORM SAMPLING FROM THE UNIT DISK

```
# uniform sampling from a unit disk centered at (0, 0)
def uniform_unit_disk():
    while True:
        # sample from the square
        (x, y) = uniform_rectangle(-1, 1, -1, 1)
        # check if (x, y) is on the disk
        if x**2 + y**2 <= 1:
            return (x, y)
```

Think: how to sample from a disk of radius r?

DARTS WITH A CIRCULAR BOARD

(Math on the board)

- Charlie throws a dart at a circular board with radius 10
- What is the probability that the distance between the hitting point and the center of the board is **at least 5**?

ASK THE AUDIENCE

- What is the probability that the dial lands on 1/8?

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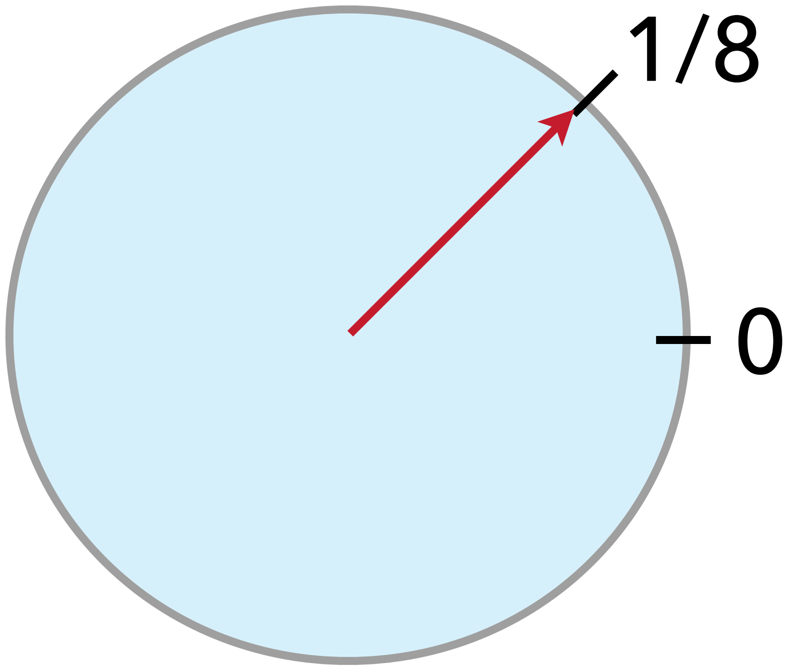
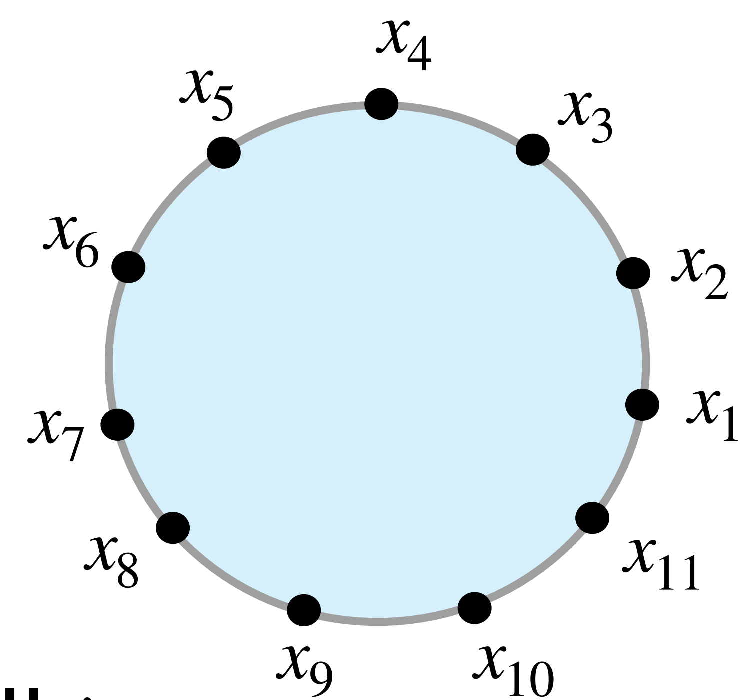
(A) 1/8	(B) 0
(C) $\pi/8$	(D) none of the above



SPINNERS

- What is the probability that the dial lands on 1/8?
- Suppose we suspect that it is, say, 0.1
- Pick 11 distinct points on the circle



by symmetry: $\Pr(\text{dial lands on } x_i) = 0.1$ for all i

by additivity: $\Pr(\text{dial lands on one of } x_1, \ldots, x_{11}) = 11 \cdot 0.1 = 1.1$

contradiction!

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SPINNERS

- Let p be the probability of an outcome, e.g., 1/8
- We can show that p = 0 by contradiction
- If $p \neq 0$: pick distinct points $x_1, \ldots, x_k$, where $k > 1/p$

by symmetry: $\Pr(\text{dial lands on } x_i) = p$ for all $i \in \{1,2,\ldots,k\}$

by additivity: $\Pr(\text{dial lands on one of } x_1, \ldots, x_k) = k \cdot p > 1$

contradiction!

SAMPLE SPACES

- Discrete sample spaces: outcomes are listable
  - We assign a probability to each outcome
  - The probability of an event is the sum of the probabilities of the outcomes comprising the event
- Continuous sample spaces: outcomes are not listable
  - Individual outcomes have probability zero
  - **T**he probability of an event is the proportion of the extent (length, area, volume, etc.) taken up by it