THE BEST WAY TO MAKE THIS DECISION IS BY CALCULATING THE EXPECTED VALUE OF EACH POSSIBLE OUTCOME.

YOU MULTIPLY THE...

MUST PRETEND TO BE DEAD.

I SENSE THAT WE'RE DONE HERE.

I HOPE THE DEAD SOMETIMES COVER THEIR EARS.
PROOF: LAW OF TOTAL EXPECTATION

- The expected value of the conditional expectation of \( X \) given \( Y \) is the same as the expected value of \( X \):

\[
\mathbb{E}(X) = \mathbb{E}[\mathbb{E}(X|Y)]
\]

**Proof:** Let \( X \) be a random variable with expected value \( \mathbb{E}(X) \) and let \( Y \) be any random variable defined on the same probability space.
PROOF: LAW OF TOTAL EXPECTATION

\[ \mathbb{E}(X) = \mathbb{E}[\mathbb{E}(X|Y)] \]
PROOF: LAW OF TOTAL EXPECTATION
FINDING THE EXPECTED VALUE WITH RECURATION

- Let $X$ be the number of trials needed to get a 6 when you throw a standard fair die.
- Let $Y$ be the number of 6s that appear on the first trial, so $Y$ is either 0 or 1.
- Then
  $$\mathbb{E}(X | Y = 1)$$
  and
  $$\mathbb{E}(X | Y = 0) = \mathbb{E}(X) + 1$$
Now notice that

\[ \mathbb{E}(X \mid Y) \begin{cases} 
1 & \text{with probability } 1/6 \\
1 + \mathbb{E}(X) & \text{with probability } 5/6
\end{cases} \]

- X is a memoryless random variable, which means it “forgets” what has come before it.
- It doesn’t matter how many times you threw the die and didn’t get a 6.
- The probability that a 6 appears in the next throw is the same.
Finding the expected value with recursion

- So

\[ E(X) = E(E(X \mid Y)) = \frac{1}{6} \cdot 1 + \frac{5}{6} \cdot (1 + E(X)) \]
VARIANCE DEFINED

- **Definition:** The variance of a random variable $X$ is

\[ \text{Var}(X) = \mathbb{E}((X - \mathbb{E}(X))^2) \]

- As in the previous example, it is helpful to read the definition from the inside out:

\[ X - \mathbb{E}(X) \rightarrow (X - \mathbb{E}(X))^2 \rightarrow \mathbb{E}((X - \mathbb{E}(X))^2) \]
Definition: The standard deviation of a random variable $X$ is

$$\sigma_X = \sqrt{\text{Var}(X)} = \sqrt{\mathbb{E}((X - \mathbb{E}(X))^2)}.$$ 

- In the previous lecture, the unit of $X$ was dollars but the unit of the variance is dollars squared.
- Since the units of variance are squared, we consider the square root of the variance.
Theorem: $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$
EXAMPLE: COIN TOSS

- Toss a coin that comes up Heads with probability $p$
- Let $X = \begin{cases} 1 & \text{if Heads} \\ 0 & \text{if Tails} \end{cases}$ and $\mathbb{E}(X) = p$
- Since $X$ is an indicator RV, we know
- $\text{Var}(X) = ?$
EXAMPLE: COIN TOSS

For \( p = \frac{1}{2} \)

\[
Var(X) = p(1 - p) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}
\]

\[
\sigma_X = \sqrt{Var(X)} = \frac{1}{2}
\]
Let $X$ be an indicator Random Variable with $\Pr(X = 1) = 1.0$

What is $\text{Var}(X)$?

A. 0

B. 1

C. Impossible to know
EXAMPLE: CONTINUOUS UNIFORM RV

- Let $X$ be a continuous RV which chooses a value from $[a,b]$ uniformly at random.

- The PDF is
  
  
  \[
  f(x) = \begin{cases} 
  \frac{1}{b-a} & \text{for } x \in [a, b] \\
  0 & \text{otherwise}
  \end{cases}
  \]

- We have
  
  \[
  \mathbb{E}(X) = \frac{a + b}{2}
  \]

Var($X$) =
COMPUTE $\mathbb{E}(X^2)$
COMPUTE $\mathbb{E}(X^2)$
PROPERTIES OF VARIANCE

- By linearity of expectation, we have

  \[ \mathbb{E}(aX + b) = a \cdot \mathbb{E}(X) + b \]

- Does this hold for variance, such that

  \[ \text{Var}(aX + b) = a \cdot \text{Var}(X) + b \]
PROPERTIES OF VARIANCE

\[ \text{Var}(aX + b) = \]
PROPERTIES OF VARIANCE

- Again, for expectation by linearity we have for any $X$ and $Y$:

$$\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y)$$

- Does this hold for variance, such that

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$$
PROPERTIES OF VARIANCE

\[ \text{Var}(X + Y) = \]