Ford-Fulkerson example - TopHat

Select *all* possible augmenting paths corresponding to the current state of the residual graph:

A. $s \rightarrow b \rightarrow d \rightarrow c \rightarrow t$

B. $s \rightarrow a \rightarrow c \rightarrow t$

C. $s \rightarrow a \rightarrow d \rightarrow t$

D. $s \rightarrow b \rightarrow a \rightarrow c \rightarrow t$

E. $s \rightarrow b \rightarrow d \rightarrow a \rightarrow c \rightarrow t$

To find augmenting paths, always use BFS.

Residual graph $G_r$: 

![Residual graph diagram]
• What is the maximum flow in the graph on the left?
• Find the max flow by running Ford-Fulkerson on the residual graph.
• Can you see some “proof” in either graph why the value of the max flow cannot be more?
st-cuts

Def. An *st-cut (cut)* is a partition \((A, B)\) of the vertices with \(s \in A\) and \(t \in B\).

\[
\begin{align*}
\text{cut: } A &= \{s, c, e, f\} \\
\text{cut-set: directed edges from nodes in } A \text{ to } B &= \{(s,a), (c,d), (f,t)\} \\
&\quad \bullet \text{ Note, that this only contains edges directed from } A \text{ to } B, \text{ not } B \text{ to } A
\end{align*}
\]
st-cuts

Def. An **st-cut (cut)** is a partition \((A, B)\) of the vertices with \(s \in A\) and \(t \in B\).

Def. The **capacity** of a cut is the sum of the capacities of the edges *from* \(A\) to \(B\).

\[
A = \{s, t\} \Rightarrow \text{cap} (\{s, t\}) = 10 + 5 + 15 = 30
\]

\[
\text{cap}(A) = \sum_{e \text{ out of } A} c(e)
\]
st-cuts

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\[A = \{s\}\]

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\[
\text{cap}(A) = \sum_{e \text{ out of } A} c(e)
\]

Note: red and green edges are not counted towards the capacity of the cut \(A\)

\(A = \{s, c, e\}\)

\(\text{cap}(A) = 10 + 8 + 16 = 34\)
**st-cuts - TopHat**

**Def.** An *st-cut (cut)* is a partition \((A, B)\) of the vertices with \(s \in A\) and \(t \in B\).

**Def.** The **capacity** of a cut is the sum of the capacities of the edges *from* \(A\) *to* \(B\).

\[
\text{cap}(A) = \sum_{e \text{ out of } A} c(e)
\]

What is the capacity of the cut given by \(A\)?

- A. 10
- B. 28
- C. 30
- D. 47

**Diagram:**

- \(A = \{s, c, e, f\}\)
- \(10 + 8 + 10 = 28\)
- **CEA**
**Minimum-cut problem**

**Def.** The capacity of a cut is the sum of the capacities of the edges from $A$ to $B$.

$$cap(A) = \sum_{e \text{ out of } A} c(e)$$

**Min st-cut problem:** Find an st-cut with minimum capacity.

\[ A = \{s, c, e, f\} \]
Minimum-cut problem

Def. The capacity of a cut is the sum of the capacities of the edges from $A$ to $B$.

$$\text{cap}(A) = \sum_{e \text{ out of } A} c(e)$$

Min st-cut problem: Find an st-cut with minimum capacity.

$$\text{cap}(A) = 28$$

$$\text{value}(f) = 28$$
Select all statements that are true.

A. The capacity of the minimum cut is unique.
B. The minimum cut is unique. ✗
C. The value of the maximum flow is unique.
D. The maximum flow is unique.
E. The value of the max flow is always equal to the capacity of the min cut.
intuition:
• any amount of flow from s to t will cross one of the edges in the min-cut
• the capacity of a cut is the “throughput” of that cut
• min-cut is the bottleneck throughput
Flow value lemma. Let $f$ be any flow and let $A$ be any cut. Then, the value of the flow $f$ equals the net flow across the cut $A$.

$$\text{val}(f) = \sum_{e \text{ leaving } A} f(e) - \sum_{e \text{ entering } A} f(e)$$

net flow across cut = $10 + 5 + 10 = 25$

$A = \{s\}$
Relationship between flows and cuts

Flow value lemma. Let $f$ be any flow and let $A$ be any cut. Then, the value of the flow $f$ equals the net flow across the cut $A$.

$$val(f) = \sum_{e \text{ leaving } A} f(e) - \sum_{e \text{ entering } A} f(e)$$

A = \{s, a, b, c, d, e, f\}

value of flow = 25
Flow value lemma. Let $f$ be any flow and let $A$ be any cut. Then, the value of the flow $f$ equals the net flow across the cut $A$.

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Relationship between flows and cuts

Flow value lemma. Let \( f \) be any flow and let \( A \) be any cut. Then, the value of the flow \( f \) equals the net flow across the cut \( A \).

\[
\text{val}(f) = \sum_{e \text{ leaving } A} f(e) - \sum_{e \text{ entering } A} f(e)
\]

Pf.

\[
\text{val}(f) = \sum_{e \text{ leaving } s} f(e) - \sum_{e \text{ entering } s} f(e) = \]

by flow conservation, all terms in the sum are 0, except for \( v = s \)

edges with both ends in \( A \) appear once with ‘+’ once with ‘-’ in the sum and cancel out. Edges with only one end in \( A \) contribute to the sum.

capacity constraint of flows

definition of \( \text{cap}(A) \)
Weak duality. Let $f$ be any flow and $A$ be any cut. Then, $v(f) \leq \text{cap}(A)$.

\[ v(f) = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) \leq \sum_{e \text{ out of } A} f(e) \leq \sum_{e \text{ out of } A} c(e) = \text{cap}(A) \]

Relationship between flows and cuts

Value of flow = 27

\[ \leq \]

Capacity of cut = 30
Certificate of optimality

**Corollary.** Let \( f \) be a flow and let \((A, B)\) be any cut. If \( \text{val}(f) = \text{cap}(A, B) \), then \( f \) is a max flow and \((A, B)\) is a min cut.

**Pf.**

- For any flow \( f' \), \( \text{val}(f') \leq \text{cap}(A) = \text{val}(f) \). \( f \) is a max flow
- For any cut \( A' \), \( \text{cap}(A') \geq \text{val}(f) = \text{cap}(A) \). \( A \) is a min cut

**Value of flow = 28** = **Capacity of cut = 28**
Relationship between flows and cuts

Max Flow Min Cut (MFMC) theorem:

Given a directed graph $G(V,E)$ with source $s$, sink $t$ and non-negative capacities $c(e)$, the value of the maximum flow in $G$ is equal to the capacity of the minimum st-cut.

\[
\max \text{ value}(f) = \min_{s \in A \subseteq V} \text{ cap}(A)
\]
Certificate of optimality

**MFMC:** the value of the max flow is equal to the capacity of the min cut

\[
\max \text{ value}(f) = \min_{s \in A \cup V} \text{ cap}(A)
\]

Certificate of optimality: We can use the MFMC theorem to prove that a flow \( f \) is maximum;
\( \text{val}(f) \) is maximum if there is a cut with its capacity equal to \( \text{val}(f) \).
Select all statements that are true. Let $G$ be a flow network and $f$ is a maximum st-flow, $A$ is a minimum capacity st-cut.

A. $f$ saturates every edge out of $s$ with flow. $\times$

B. every edge from $A$ to $V-A$ is saturated with flow. $\checkmark$

C. every edge from $V-A$ to $A$ has 0 flow. $\checkmark$

D. If we increase the capacity of each edge in $A$ by 1, then the value of the max flow will increase. $\times$
Certificate for the max flow

• Find the minimum cut (look at the max flow to find it)

• How does the residual graph help in finding it?
network $G$ and flow $f$

cut $A = \{s, v\}$
cap($A$) = 10 + 9 = 19

val($f$) = 19

claim: $A$ is a min-cut.

residual network $G_f$

only $v$ is reachable from $s$ in $G_f$
Finding the min-cut

1. find the maximum flow in $G$, i.e. run Ford-Fulkerson
2. find the set $A$ of all nodes that are still reachable from $s$ in $G_f$
   • run BFS from $s$ in $G_f$ to find $A$
   • $A$ has at least one element, $s$

Nodes in $A$ form the minimum capacity cut.
Properties of min-cuts

1. is the min-cut always unique?

While the min-cut value is always unique, there can be multiple different cuts that achieve the same minimum cut value.

2. what happens to the max flow if we decrease the capacity of an edge in the min-cut by 1?

There are two scenarios
- The min-cut is not unique, then the flow does not change
- The min-cut is unique; then the flow might change
Certificate of optimality

**MFMC**: the value of the max flow is equal to the capacity of the min cut

\[
\max \text{ value}(f) = \min_{s \in A \subseteq V} \text{ cap}(A)
\]

Certificate of optimality: We can use the MFMC theorem to prove that a flow \( f \) is maximum;
\( \text{val}(f) \) is maximum if there is a cut with its capacity equal to \( \text{val}(f) \).

![Graph with values](image)

value of flow = 28

= capacity of cut = 28
Certificate for the max flow - finding the min cut

- When running F-F, how do we know that we have found the max flow?
- How does this help identifying the minimum cut?
Certificate for the max flow - finding the min cut

network G and flow f

cut $A = \{s, v\}$
cap($A$) = 10 + 9 = 19

$\text{val}(f) = 19$

claim: $A$ is a min-cut.

residual network $G_f$

only $v$ is reachable from $s$ in $G_f$
Finding the min-cut

1. find the maximum flow in $G$, i.e. run Ford-Fulkerson
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   • run BFS from $s$ in $G_f$ to find $A$
   • $A$ has at least one element, $s$

Nodes in $A$ form the minimum capacity cut.
The edges highlighted in green form a BFS tree from s in the residual graph. Identify the min-cut corresponding to it.

Min cut set A =
A. \{ s, b, c\}
B. \{s, a, b, c, d, e\}
C. \{s, a\}
D. \{s, a, b, c, d\}

(Reversed edges in the residual are dashed lines only for purpose of better visibility.)
**Problem:** We are given a flow network — a directed graph $G(V,E)$ with source $s$, sink $t$ and edge capacities $c(e)$. Further, for every vertex $v$ we are given a capacity $d(v)$, a limit on the amount of flow that can pass through it. Find the maximum $st$-flow given the edge and vertex capacities.

**Throughput limit on vertices**

- $d(a) = 20$
- $d(b) = 18$

Maximum flow without vertex throughput limit = 33
Bipartite matching

**Def.** A graph $G$ is **bipartite** if the nodes can be partitioned into two subsets $L$ and $R$ such that every edge connects a node in $L$ to one in $R$.

**Matching.** Assign each node to at most one of its neighbors.

**Problem.** Given a bipartite graph $G = (L \cup R, E)$, find a max size matching.
Bipartite matching

Def. A graph $G$ is bipartite if the nodes can be partitioned into two subsets $L$ and $R$ such that every edge connects a node in $L$ to one in $R$.

Matching. Assign each node to at most one of its neighbors.

Problem. Given a bipartite graph $G = (L \cup R, E)$, find a max size matching.

Diagram:

```
  1  --  1'
    |    |
  2  --  2'
    |    |
  3  --  3'
    |    |
  4  --  4'
    |    |
  5  --  5'
```

Matching: 1–1', 2–2', 3–4', 4–5'

3' and 4' share only one neighbor hence, only one can be matched.
Bipartite matching: max-flow formulation

- Create directed graph $G' = (L \cup R \cup \{s, t\}, E')$.
- Direct all edges from $L$ to $R$, and assign unit capacity.
- Add source $s$ and unit-capacity edges from $s$ to each node in $L$.
- Add sink $t$, and unit-capacity edges from each node in $R$ to $t$. 

![Graph Image]
Bipartite matching vs. Max-flow

There is a one-to-one correspondence between the edges in the bipartite matching and the augmenting paths in the flow network.

\[
\min \text{ cut } A = \{1, 3, 5, 2'\}
\]
Select all true statements.

A. The value of the maximum flow = size of the maximal matching
B. The min cut always consists of s and the nodes in the L set, e.g. \{s,1,2,3,4,5\}  
C. At most one unit can flow through each node in L as their total in-capacity is 1.
D. At most one unit can flow through each node as all edge capacities are 1.

\[
\text{min-cut} = \{s,5,2'\}
\]