BU CAS Mathematics 142 (Spring, 2012)
Introduction to Linear Algebra

Practice Final Exam
Linear Algebra

Thursday, May 3, 2012

You will have 120 minutes to complete the final exam. You will be permitted to use a two-sided 8.5 in. × 11 in. sheet filled with any information you choose to include.

Your solutions must show all work and include justifications.

NAME	

Problem #1		*10*
Problem #2		*10*
Problem #3		*10*
Problem #4		*10*
Problem #5		*20*
Problem #6		*20*
Problem #7		*20*
Total		*100*

Problem #8		*10*
Extra Credit		*10*

Problem 1. (10 pts.)

Let $\left\{ \begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix} \right\}$ be an orthonormal basis. If $a = \frac{\sqrt{3}}{2}$ and $c + d > \frac{\sqrt{3}-1}{2}$, find b, c, and d.

We know that the following are true:

$$\begin{aligned} \begin{bmatrix} a \\ c \end{bmatrix} \cdot \begin{bmatrix} b \\ d \end{bmatrix} &= 0 \\ \left|\left| \begin{bmatrix} a \\ c \end{bmatrix} \right|\right| &= 1 \\ \left|\left| \begin{bmatrix} b \\ d \end{bmatrix} \right|\right| &= 1 \end{aligned}$$

Thus, we have:

$$\begin{aligned} ab + cd &= 0 \\ a^2 + c^2 &= 1 \\ b^2 + d^2 &= 1 \end{aligned}$$

This implies that:

$$\begin{aligned} c &= 1 - \left(\frac{\sqrt{3}}{2}\right)^2 \\ &= \pm\frac{1}{2} \end{aligned}$$

Since we must have $c + d > 0$, we let $c = \frac{1}{2}$, so:

$$\begin{aligned} \frac{\sqrt{3}}{2}b + \frac{1}{2}d &= 0 \\ b &= \pm\frac{1}{2} \\ d &= \pm\frac{\sqrt{3}}{2} \end{aligned}$$

Since we want $c + d > 0$, we let $d = \frac{\sqrt{3}}{2}$ and $b = -\frac{\sqrt{3}}{2}$.

Problem 2. (10 pts.)

Find two unit vectors orthogonal to both $\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$.

It is sufficient to find the one-dimensional space that is the orthogonal complement of $\text{span}\left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$, and then to find the two unit vectors in that space.

Any vector in the orthogonal complement is a solution to the system:

$$\begin{aligned} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} &= 0 \\ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} &= 0 \end{aligned}$$

This underdetermined system can be rewritten in terms of a single variable x:

$$\begin{aligned} x + 3y - z &= 0 \\ -2x + y &= 0 \\ y &= 2x \\ z &= x + 3y = x + 6x = 7x \end{aligned}$$

Thus, if we set $x = 1$, the orthogonal complement is:

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}^{\perp} = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} \right\}.$$

Since $\left|\left| \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} \right|\right| = 3\sqrt{6}$, the two unit vectors are:

$$\begin{bmatrix} \frac{1}{3\sqrt{6}} \\ \frac{2}{3\sqrt{6}} \\ \frac{7}{3\sqrt{6}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{3\sqrt{6}} \\ -\frac{2}{3\sqrt{6}} \\ -\frac{7}{3\sqrt{6}} \end{bmatrix}.$$

Problem 3. (10 pts.)

Let M be a matrix and let $f(v) = M \cdot v$ be a linear transformation. For each part below, find any definition for M that satisfies the specified property of f.

(a) Find any M such that $f \in \mathbb{R}^2 \to \mathbb{R}^3$ and $\mathsf{dim}(\mathsf{im}(f)) = 1$.

It is sufficient to find any matrix $M \in \mathbb{R}^{3\times 2}$ that has two linearly dependent column vectors. For example:

$$M = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix}.$$

(b) Find any M such that $f \in \mathbb{R}^3 \to \mathbb{R}^3$ and $\mathsf{im}(f) = \mathsf{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$.

It is sufficient to find any matrix $M \in \mathbb{R}^{3\times 3}$ that has three linearly dependent column vectors in the specified span. For example:

$$M = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 2 & 2 & 2 \end{bmatrix}.$$

(c) Find any M such that $(\mathsf{ker}(f))^\perp = \mathsf{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}$.

If $g(v) = M^\top v$, then $(\mathsf{ker}(f))^\perp = \mathsf{im}(g)$, so we can simply let:

$$M^\top = \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ 3 & 2 \end{bmatrix} \qquad M = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 0 & 2 \end{bmatrix}.$$

(d) Find any M such that f is surjective but *not* injective. You must show that f is surjective and provide two explicit inputs that show it is not injective.

It is sufficient to find any matrix M that maps a vector space V to a vector space W such that $\mathsf{dim}(V) > \mathsf{dim}(W)$ but $\mathsf{im}(f) = W$. For example, let $V = \mathbb{R}^3$ and $W = \mathbb{R}^2$:

$$M = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \qquad \mathsf{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\} = \mathbb{R}^2$$

Then, the following two vectors both map to the same vector:

$$M \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad M \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Problem 4. (10 pts.)

Consider the following vector space:

$$V = \{M \mid M \in \mathbb{R}^{2\times 2}\}.$$

Let $f : V \to V$ be a map defined by:

$$f(M) = M^\top.$$

Show that f is a linear transformation (i.e., that f satisfies all the properties of a linear transformation).

It is sufficient to show that f maps $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ to $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, that it preserves vector addition, and that it preserves scalar multiplication.

We have that:

$$\begin{aligned} f\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}^\top \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

We also have for any $A, B \in \mathbb{R}^{2\times 2}$ that:

$$\begin{aligned} f(A+B) &= (A+B)^\top \\ &= A^\top + B^\top \\ &= f(A) + f(B) \end{aligned}$$

Finally, we have for any $A \in \mathbb{R}^{2\times 2}$ and $s \in \mathbb{R}$ that:

$$\begin{aligned} f(s \cdot A) &= (s \cdot A)^\top \\ &= s \cdot (A^\top) \\ &= s \cdot f(A) \end{aligned}$$

Problem 5. (20 pts.)

Consider the following vector subspaces of $\mathbb{R}^3$:

$$V = \mathsf{span}\left\{\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}\right\} \qquad W = \mathsf{span}\left\{\begin{bmatrix} -4 \\ 1 \\ 2 \end{bmatrix}\right\}$$

You are in a spaceship positioned at $\begin{bmatrix} 20 \\ 15 \\ -30 \end{bmatrix}$ and a signal transmitter is positioned at $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(a) Suppose the transmitter's signal beam only travels in two opposite directions along V. What is the shortest distance your spaceship must travel to intercept the signal beam?

The closest interception point is the projection of the ship's position onto V. First, we find the unit vector that spans V:

$$u = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ \frac{-2}{\sqrt{5}} \end{bmatrix}.$$

Next, we project the spaceship's position onto $\mathsf{span} u$ using the formula $(u \cdot v) \cdot u$:

$$p = \left(\begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ \frac{-2}{\sqrt{5}} \end{bmatrix} \cdot \begin{bmatrix} 20 \\ 15 \\ -30 \end{bmatrix}\right) \cdot \begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ \frac{-2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 16 \\ 0 \\ -32 \end{bmatrix}.$$

Finally, the distance is

$$\left|\left|\begin{bmatrix} 20 \\ 15 \\ -30 \end{bmatrix} - \begin{bmatrix} 16 \\ 0 \\ -32 \end{bmatrix}\right|\right| = \left|\left|\begin{bmatrix} 4 \\ 15 \\ 2 \end{bmatrix}\right|\right| = \sqrt{245} = 7\sqrt{5}$$

(b) Suppose the transmitter is also rotating around the axis collinear with W. What is the shortest distance your spaceship must travel to reach a position at which it can intercept the signal beam?

Because the transmitter is rotating, the beam sweeps around and covers a plane that is perpendicular to the axis of rotation. The closest interception point is the projection of the ship's position onto $W^{\perp}$. We have that $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ is orthogonal to W. We find another vector orthogonal to W and $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$:

$$\begin{aligned} -4x + y + 2z &= 0 \\ x - 2z &= 0 \\ z &= \frac{1}{2}x \\ y &= 3x \end{aligned}$$

Let $x = 2$. Thus, we want to project the spaceship's position onto the image of:

$$M = \begin{bmatrix} 1 & 2 \\ 0 & 6 \\ -2 & 1 \end{bmatrix}.$$

The projection is then:

$$\begin{aligned} p &= M \cdot (M^{\top} \cdot M)^{-1} \cdot M^{\top} \cdot \begin{bmatrix} 20 \\ 15 \\ -30 \end{bmatrix} \\ &= M \cdot \frac{1}{205} \cdot \begin{bmatrix} 41 & 0 \\ 0 & 5 \end{bmatrix} \cdot M^{\top} \cdot \begin{bmatrix} 20 \\ 15 \\ -30 \end{bmatrix} \\ &= \frac{1}{205} \cdot \begin{bmatrix} 4280 \\ 3000 \\ -6060 \end{bmatrix} \end{aligned}$$

As in part (a), the distance is $\left|\left| \begin{bmatrix} 20 \\ 15 \\ -30 \end{bmatrix} - \frac{1}{205} \cdot \begin{bmatrix} 4280 \\ 3000 \\ -6060 \end{bmatrix} \right|\right| \approx \begin{bmatrix} -0.9 \\ 0.4 \\ -0.4 \end{bmatrix}$.

Problem 6. (20 pts.)

You are given the following data points:

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Find a polynomial of the form $f(x) = ax + b$ that is the least squares best fit for this data.
Hint: use the formula for computing a projection onto the image of a matrix.

We are looking for an approximate solution to the overdetermined system:

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}.$$

We project $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ onto the image of the matrix above:

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \cdot \left(\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}^{\top} \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \right)^{-1} \cdot \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}^{\top} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 2 \\ 5/2 \end{bmatrix}.$$

Now, if we replace $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ with its projection $\begin{bmatrix} 3/2 \\ 2 \\ 5/2 \end{bmatrix}$, the system is no longer overdetermined:

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3/2 \\ 2 \\ 5/2 \end{bmatrix}.$$

The approximate solution is then:

$$f(x) = \frac{1}{2}x + 2.$$

Problem 7. (20 pts.)

Suppose that vectors in $\mathbb{R}^2$ represent population quantities in two locations, f represents a change in the quantities over the course of one year if the economy is doing well, g represents a change in the quantities over the course of one year if the economy is not doing well, and w is the initial state:

$$f(v) = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \cdot v \qquad g(v) = \begin{bmatrix} 0.1 & 0.2 \\ 0.4 & 0.3 \end{bmatrix} \cdot v \qquad w = \begin{bmatrix} 100 \\ 200 \end{bmatrix}$$

If the population state is initially w, find the population state after 100 years of a good economy and 202 years of a poor economy.

Notice that for $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $f\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = 4 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $g\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \frac{1}{2} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Notice that this is also true for any scalar multiple of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (we demonstrate only the case for f below):

$$\begin{aligned} f(s \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}) &= \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \cdot (s \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}) \\ &= s \cdot (\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}) \\ &= s \cdot (4 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}) \end{aligned}$$

Thus, after 100 years of a good economy and 202 of a poor economy, we have:

$$\begin{aligned} 4^{100} \cdot \left(\frac{1}{2}\right)^{202} \cdot \begin{bmatrix} 100 \\ 200 \end{bmatrix} &= 2^{200} \cdot 2^{-202} \cdot \begin{bmatrix} 100 \\ 200 \end{bmatrix} \\ &= 2^{200} \cdot 2^{-202} \cdot \begin{bmatrix} 100 \\ 200 \end{bmatrix} \\ &= 2^{-2} \cdot \begin{bmatrix} 100 \\ 200 \end{bmatrix} \\ &= \begin{bmatrix} 25 \\ 50 \end{bmatrix} \end{aligned}$$

Technically, 4 is an eigenvalue of f with eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\frac{1}{2}$ is an eigenvalue of g with eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. However, it is only necessary to realize that f and g behave like scalar multiplies on the space $\mathsf{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$.

Problem 8. (10 extra credit pts.)

Consider the following set of linear transformations:

$$\mathcal{T} = \{f \mid f : \mathbb{R}^2 \to \mathbb{R}^2,\ f \text{ is a linear transformation},\ f\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\}.$$

We introduce the following definitions; let $f, g \in \mathcal{T}$ and $s \in \mathbb{R}$:

$$\begin{aligned} \mathbb{O} &= h \quad \text{where } h(v) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ f \oplus g &= h \quad \text{where } h(v) = f(v) + g(v) \\ s \otimes f &= h \quad \text{where } h(v) = s \cdot f(v) \end{aligned}$$

(a) Show that with the additive identity $\mathbb{O}$, vector addition operation $\oplus$, and scalar multiplication operation $\otimes$ defined above, $\mathcal{T}$ is a vector space (it is sufficient to show that the appropriate membership and closure properties are satisfied).

(b) Show that $\phi : \mathcal{T} \to \mathbb{R}^2$ is a linear transformation if it is defined as:

$$\phi(f) = f\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right).$$