

# BU CS 332 – Theory of Computation

Link to polls: <https://forms.gle/2hWWjEjNge1P1V9J9>

## Lecture 2:

- Parts of a Theory of Computation
- Sets, Strings, and Languages

Reading:  
Sipser Ch 0

Mark Bun

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# What makes a good theory?

- General ideas that apply to many different systems
- Expressed simply, abstractly, and precisely

## Parts of a Theory of Computation

- Models for **machines** (computational devices)
- Models for the **problems** machines can be used to solve
- **Theorems** about what kinds of machines can solve what kinds of problems, and at what cost

# What is a (Computational) Problem?

For us: A problem will be the task of **recognizing whether a string is in a language**

- **Alphabet:** A finite set  $\Sigma$                       Ex.  $\Sigma = \{a, b\}$
- **String:** A finite concatenation of alphabet symbols  
Ex.  $bba, ababb$   
 $\varepsilon$  denotes empty string, length 0  
 $\Sigma^*$  = set of all strings using symbols from  $\Sigma$   
Ex.  $\{a, b\}^* = \{\varepsilon, a, b, aa, ab, ba, bb, \dots\}$
- **Language:** A set  $L \subseteq \Sigma^*$  of strings

# Examples of Languages

**Parity:** Given a string consisting of a's and b's, does it contain an even number of a's?

$\Sigma =$                        $L =$

**Primality:** Given a natural number  $x$  (represented in binary), is  $x$  prime?

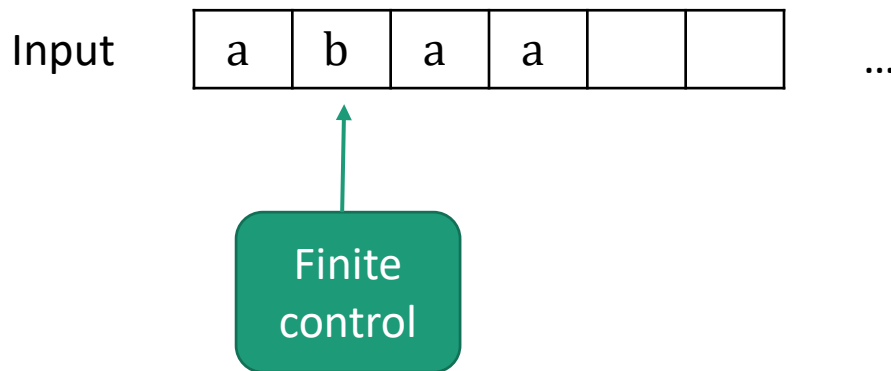
$\Sigma =$                        $L =$

**Halting Problem:** Given a C program, can it ever get stuck in an infinite loop?

$\Sigma =$                        $L =$

# Machine Models

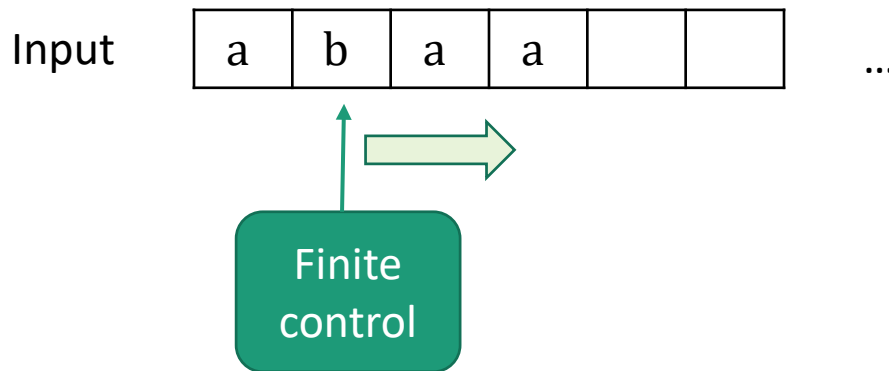
Computation is the processing of information by the **unlimited application** of a **finite set** of operations or rules



Abstraction: We don't care how the control is implemented. We just require it to have a finite number of states, and to transition between states using fixed rules.

# Machine Models

- Finite Automata (FAs): Machine with a finite amount of unstructured memory

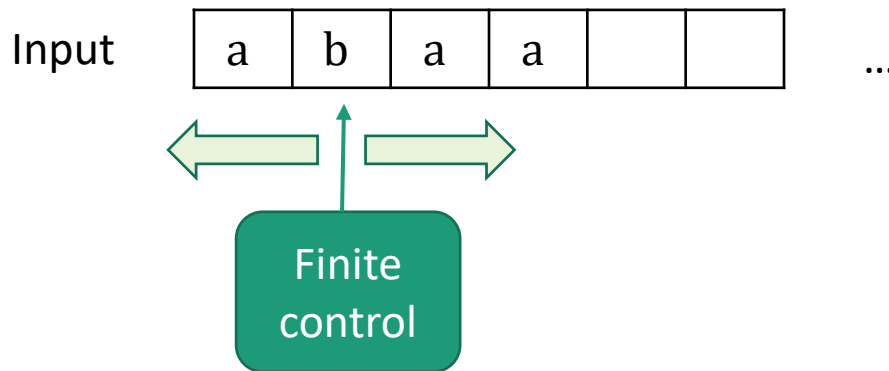


Control scans left-to-right  
Can check simple patterns  
Can't perform unlimited counting

Useful for modeling chips, simple control systems, choose-your-own adventure games...

# Machine Models

- Turing Machines (TMs): Machine with unbounded, unstructured memory



Control can scan in both directions  
Control can both read and write

Model for general sequential computation

**Church-Turing Thesis:** Everything we intuitively think of as “computable” is computable by a Turing Machine

# What theorems would we like to prove?

We will define classes of languages based on which machines can recognize them

**Inclusion:** Every language recognizable by a FA is also recognizable by a TM

**Non-inclusion:** There exist languages recognizable by TMs which are not recognizable by FAs

**Completeness:** Identify a “hardest” language in a class

**Robustness:** Alternative definitions of the same class

Ex. Languages recognizable by FAs = regular expressions



# Why study theory of computation?

- You'll learn how to formally reason about computation
- You'll learn the technology-independent foundations of CS

## Philosophically interesting questions:

- Are there well-defined problems which cannot be solved by computers?
- Can we always find the solution to a puzzle faster than trying all possibilities?
- Can we say what it means for one problem to be “harder” than another?

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## Connections to other parts of science:

- Finite automata arise in compilers, AI, coding, chemistry  
<https://cstheory.stackexchange.com/a/14818>
- Hard problems are essential to cryptography
- Computation occurs in cells/DNA, the brain, economic systems, physical systems, social networks, etc.

# What appeals to you about the theory of computation?



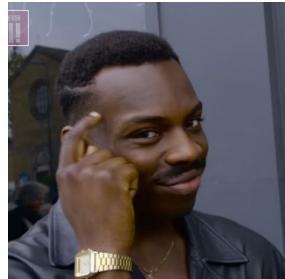
1. I want to learn new ways of thinking about computation
2. I like math and want to see how it's used in computer science
3. I'm excited about the philosophical questions about computation
4. I want to practice problem solving and algorithmic thinking
5. I want to develop a "computational perspective" on other areas of math/science
6. I actually wanted to take CS 320 or 350 but they were full

# Why study theory of computation?

## Practical knowledge for developers



“Boss, I can’t find an efficient algorithm.  
I guess I’m just too dumb.”



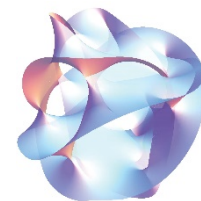
“Boss, I can’t find an efficient algorithm  
because no such algorithm exists.”

Will you be asked about this material on job interviews?

No promises, but a true story...

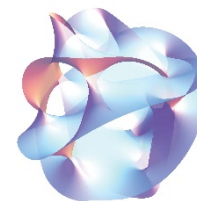
# More about strings and languages

# String Theory



- **Symbol:** Ex. a, b, 0, 1
- **Alphabet:** A finite set  $\Sigma$  Ex.  $\Sigma = \{a, b\}$
- **String:** A finite concatenation of alphabet symbols  
Ex. bba, ababb  
 $\varepsilon$  denotes empty string, length 0  
 $\Sigma^*$  = set of all strings using symbols from  $\Sigma$   
Ex.  $\{a, b\}^* = \{\varepsilon, a, b, aa, ab, ba, bb, \dots\}$
- **Language:** A set  $L \subseteq \Sigma^*$  of strings

# String Theory



- **Length** of a string, written  $|x|$ , is the number of symbols

Ex.  $|abba| =$                        $|\varepsilon| =$

- **Concatenation** of strings  $x$  and  $y$ , written  $xy$ , is the symbols from  $x$  followed by the symbols from  $y$

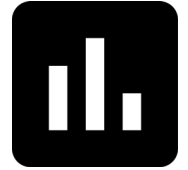
Ex.  $x = ab, y = ba \quad \Rightarrow \quad xy =$

$x = ab, y = \varepsilon \quad \Rightarrow \quad xy =$

- **Reversal** of string  $x$ , written  $x^R$ , consists of the symbols of  $x$  written backwards

Ex.  $x = aab \quad \Rightarrow \quad x^R =$

# Fun with String Operations



What is  $(xy)^R$ ?

Ex.  $x = aba, y = bba \Rightarrow xy =$   
 $\Rightarrow (xy)^R =$

- a)  $x^R y^R$
- b)  $y^R x^R$
- c)  $(yx)^R$
- d)  $xy^R$



# Fun <sup>Proofs</sup> with String Operations

**Claim:**  $(xy)^R =$

**Proof:** Let  $x = x_1x_2 \dots x_n$  and  $y = y_1y_2 \dots y_m$   
Then  $(xy)^R =$

Not even the most formal way to do this:

1. Define string length recursively
2. Prove by induction on  $|y|$

# Languages

A language  $L$  is a set of strings over an alphabet  $\Sigma$

i.e.,  $L \subseteq \Sigma^*$

Languages = computational (decision) problems

Input: String  $x \in \Sigma^*$

Output: Is  $x \in L$ ? (YES or NO?)

# Some Simple Languages

$$\Sigma = \{0, 1\}$$

$$\Sigma = \{a, b, c\}$$

$\emptyset$  (Empty set)

$\Sigma^*$  (All strings)

$\Sigma^n = \{x \in \Sigma^* \mid |x| = n\}$   
(All strings of length  $n$ )

# Some More Interesting Languages

- $L_1 =$  The set of strings  $x \in \{a, b\}^*$  that have an equal number of a's and b's
- $L_2 =$  The set of strings  $x \in \{a, b\}^*$  that start with (0 or more) a's and are followed by an equal number of b's
- $L_3 =$  The set of strings  $x \in \{0, 1\}^*$  that contain the substring '0100'

# Some More Interesting Languages

- $L_4$  = The set of strings  $x \in \{a, b\}^*$  of length at most 4
- $L_5$  = The set of strings  $x \in \{a, b\}^*$  that contain at least two a's

# New Languages from Old

$L_6$  = The set of strings  $x \in \{a, b\}^*$  that have an equal number of a's and b's and length greater than 4

Since languages are just sets of strings, can build them using set operations:

$A \cup B$       “union”

$A \cap B$       “intersection”

$\bar{A}$       “complement”

# New Languages from Old

$L_6$  = The set of strings  $x \in \{a, b\}^*$  that have an equal number of a's and b's and have length greater than 4

- $L_1$  = The set of strings  $x \in \{a, b\}^*$  that have an equal number of a's and b's
- $L_4$  = The set of strings  $x \in \{a, b\}^*$  of length at most 4

$$\Rightarrow L_6 =$$

# Operations Specific to Languages

- **Reverse:**  $L^R = \{x^R \mid x \in L\}$

Ex.  $L = \{\varepsilon, a, ab, aab\} \Rightarrow L^R =$

- **Concatenation:**  $L_1 \circ L_2 = \{xy \mid x \in L_1, y \in L_2\}$

Ex.  $L_1 = \{ab, aab\} \quad L_2 = \{\varepsilon, b, bb\}$

$\Rightarrow L_1 \circ L_2 =$



# A Few “Traps”

String, language, or something else?



$\varepsilon$

$\emptyset$

$\{\varepsilon\}$

$\{\emptyset\}$

# Languages

Languages = computational (decision) problems

Input: String  $x \in \Sigma^*$

Output: Is  $x \in L$ ? (YES or NO?)

The language **recognized** by a program is the set of strings  $x \in \Sigma^*$  that it *accepts*

# What Language Does This Program Recognize?



Alphabet  $\Sigma = \{a, b\}$

On input  $x = x_1x_2 \dots x_n$ :

count = 0

For  $i = 1, \dots, n$ :

If  $x_i = a$ :

count = count + 1

If count  $\leq 4$ : **accept**

Else: **reject**

- a)  $\{x \in \Sigma^* \mid |x| > 4\}$
- b)  $\{x \in \Sigma^* \mid |x| \leq 4\}$
- c)  $\{x \in \Sigma^* \mid |x| = 4\}$
- d)  $\{x \in \Sigma^* \mid x \text{ has more than 4 a's}\}$
- e)  $\{x \in \Sigma^* \mid x \text{ has at most 4 a's}\}$
- f)  $\{x \in \Sigma^* \mid x \text{ has exactly 4 a's}\}$