BU CS 332 – Theory of Computation

https://forms.gle/nhDVMnUWLYLjdYZ2A

Lecture 5:

- Closure Properties
- Regular Expressions

Reading:

Sipser Ch 1.2-1.3

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Last Time

- NFAs vs. DFAs
  - Subset construction: NFA -> DFA

- Intro to closure properties of regular languages

Closure Properties

Operations on languages

Let $A, B \subseteq \Sigma^*$ be languages. Define

Regular Operations:

- Union: $A \cup B$ $= \{x \mid x \in A \text{ or } x \in B\}$
- Concatenation: $A \circ B = \{xy \mid x \in A, y \in B\}$
- Star: $A^* = \{ w_1 w_2 \ldots w_n \mid n \geq 0 \text{ and } w_i \in A\}$
  $= \{\varepsilon\} \cup A \cup A \circ A \cup A \circ A \circ A \cup \ldots$

Complement: $\bar{A}$

Intersection: $A \cap B$

Reverse: $A^R = \{ a_1 a_2 \ldots a_n \mid a_n \ldots a_1 \in A\}$

Theorem: The class of regular languages is closed under all six of these operations (i.e., if A and B are both regular, all of the above langs are regular)

Proving Closure Properties

Complement

N accept w iff
M_A rejects w

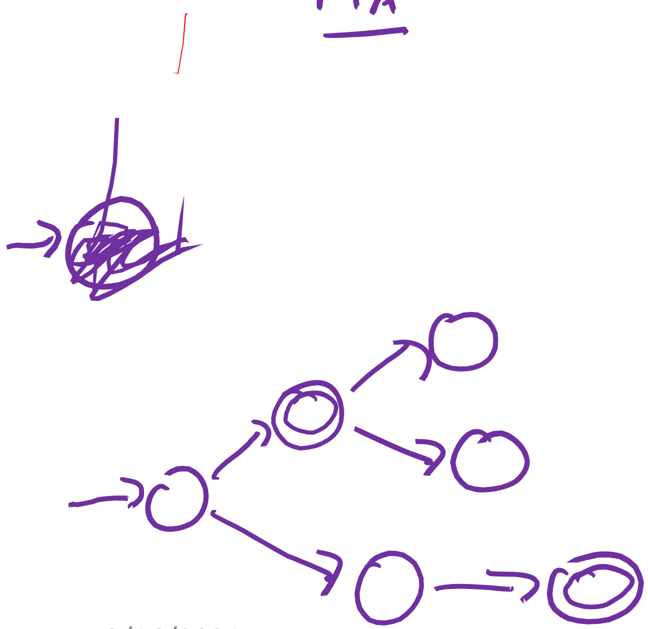
Complement: $\bar{A} = \{ w \mid w \notin A \}$

Theorem: If A is regular, then $\bar{A}$ is also regular

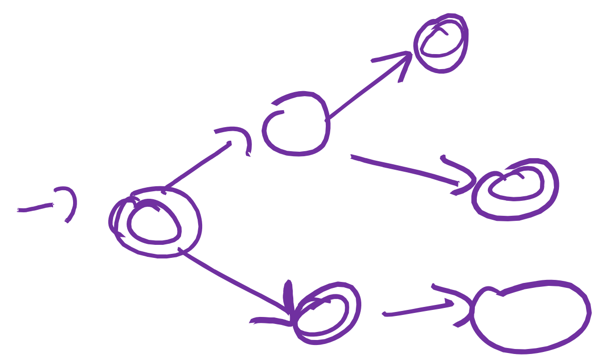
Proof idea: A regular ⇒ ∃ a DFA M_A recognizes A

Goal: construct new DFA N s.t. N recognizes $\bar{A}$

M_A



N



Complement, Formally

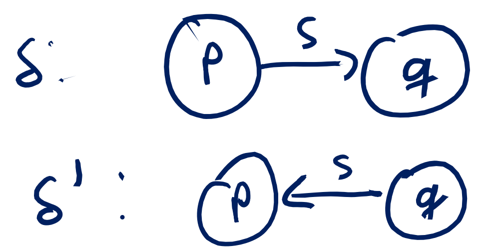
Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA recognizing a language A. Which of the following represents a DFA recognizing $\bar{A}$?

a) $(F, \Sigma, \delta, q_0, Q)$

b) $(Q, \Sigma, \delta, q_0, Q \setminus F)$, where $Q \setminus F$ is the set of states in Q that are not in F

c) $(Q, \Sigma, \delta', q_0, F)$ where $\delta'(q, s) = p$ such that $\delta(p, s) = q$

d) None of the above



↑ "almost" recognizes A^R

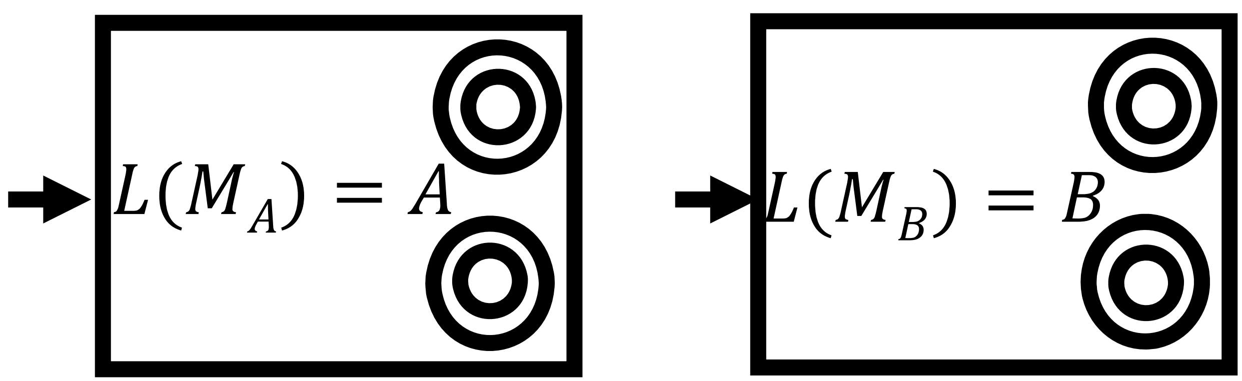
Closure under Concatenation

Concatenation: $A \circ B \ = \ \{ \, xy \ | x \in A, y \in B \ \}$

Theorem. If A and B are regular, $A \circ B$ is also regular.

Proof idea: Given DFAs M_A and M_B, construct NFA by

- Connecting all accept states in M_A to the start state in M_B.
- Make all states in M_A non-accepting.



Closure under Concatenation

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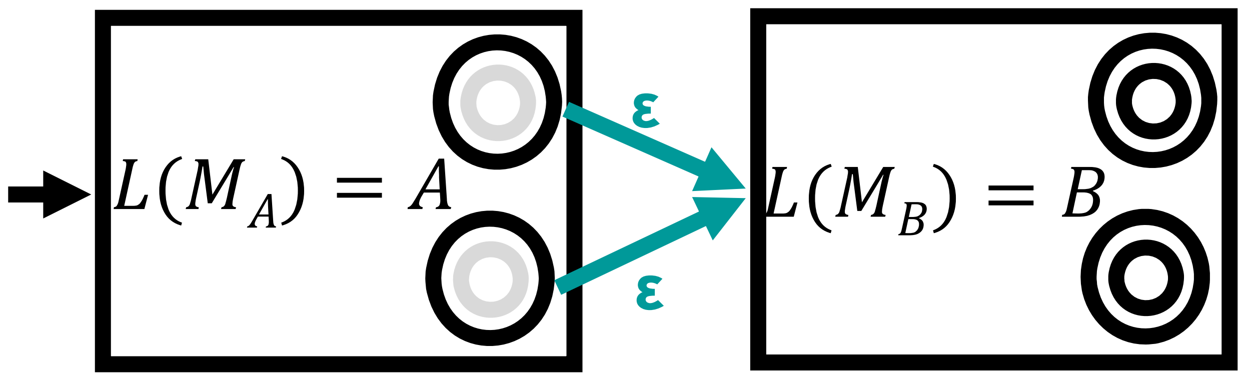
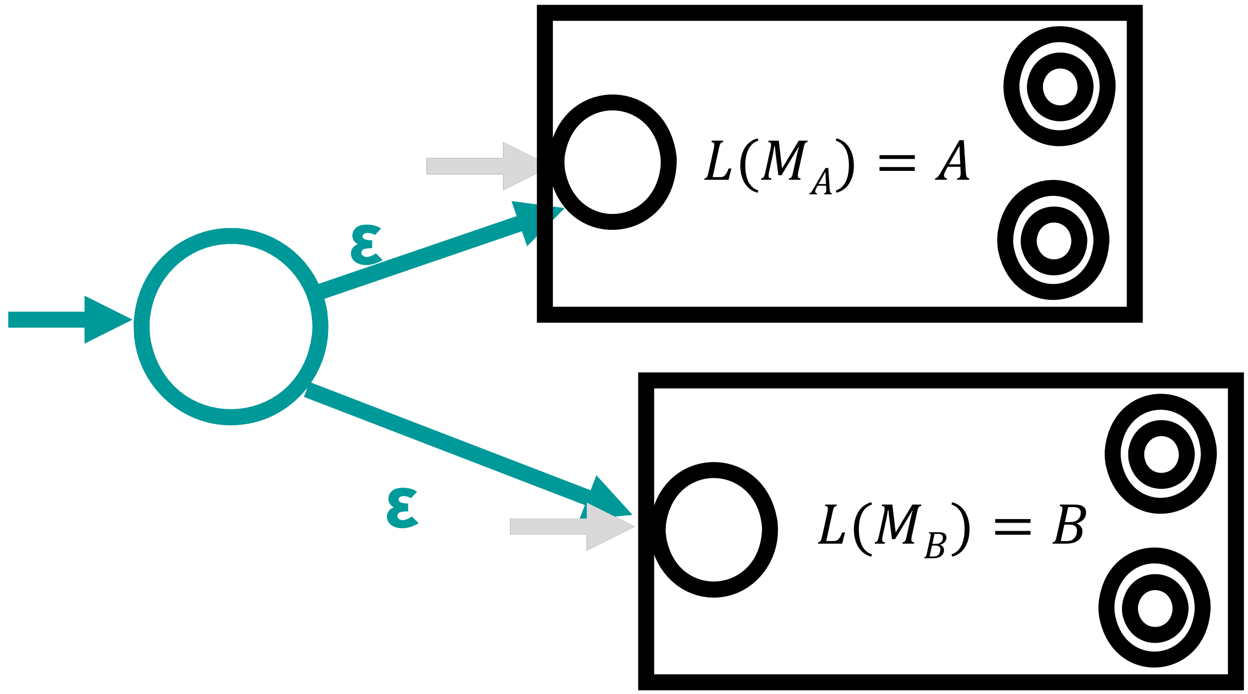


A Mystery Construction

Given DFAs M_A recognizing A and M_B recognizing B, what does the following NFA recognize?

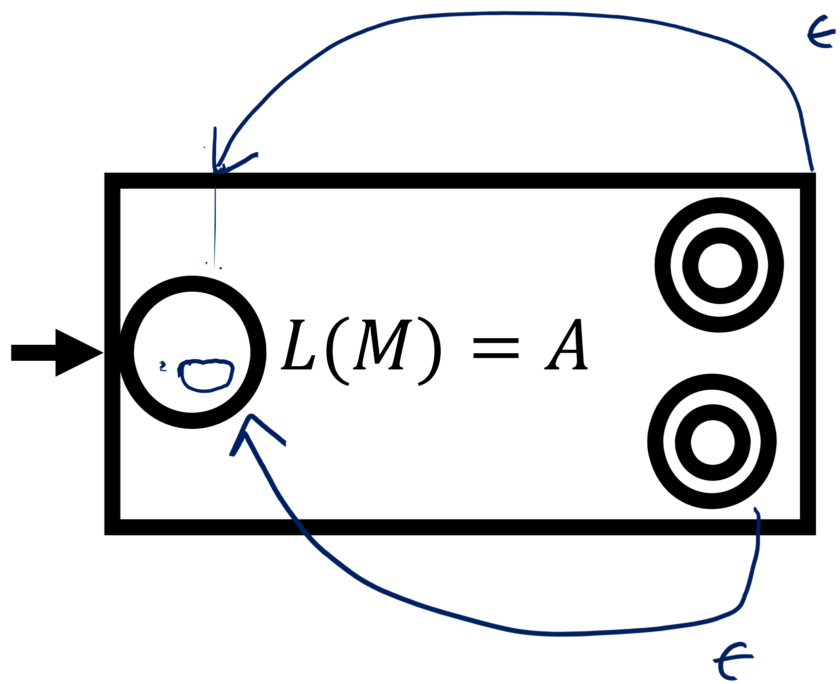


a) $A \cup B$
b) $A \circ B$
c) $A \cap B$
d) $\{\varepsilon\} \cup A \cup B$

Closure under Star

Star: $A^* = \{ a_1 a_2 \ldots a_n \mid n \geq 0 \text{ and } a_i \in A \}$

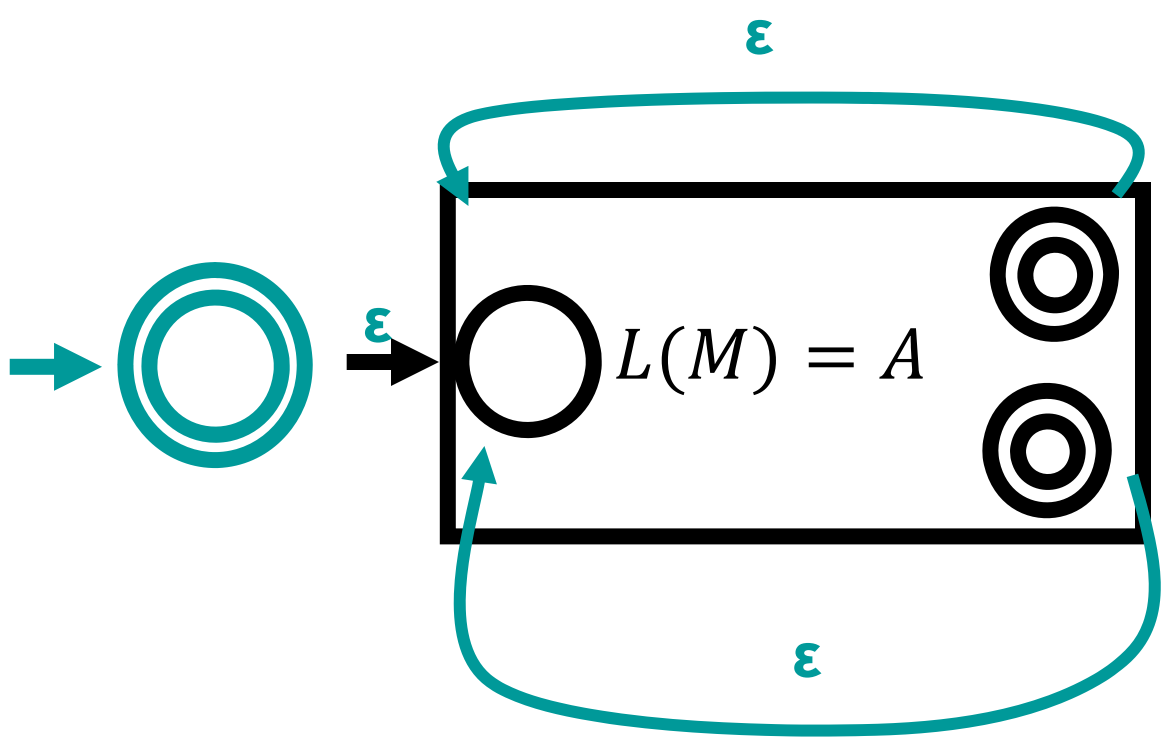
Theorem. If A is regular, A^* is also regular.



Closure under Star

Star: $A^* = \{ a_1 a_2 ... a_n \,|\, n \geq 0 \text{ and } a_i \in A\}$

Theorem. If A is regular, A^* is also regular.



On proving your own closure properties

You'll have homework/test problems of the form "show that the regular languages are closed under some operation"

Given op(A, B), show that if A, B are regular, then op(A, B) is also regular

What would Sipser do?

- Give the "proof idea": Explain how to take machine(s) recognizing regular language(s) and create a new machine
- Explain in a few sentences why the construction works
- Give a formal description of the construction
- No need to formally prove that the construction works

Regular Expressions

Regular Expressions

- A different way of describing regular languages
- A regular expression expresses a (possibly complex) language by combining simple languages using the regular operations

"Simple" languages: $\emptyset, \{\varepsilon\}, \{a\}$ for some $a \in \Sigma$

Regular operations:

Union: $A \cup B$

Concatenation: $A \circ B = \{ab \mid a \in A, b \in B\}$

Star: $A^* = \{ a_1 a_2 \ldots a_n \mid n \geq 0$ and $a_i \in A\}$

Regular Expressions – Syntax

$(a \circ a)$

A regular expression R is defined recursively using the following rules:

1. ε, $\emptyset$, and a are regular expressions for every $a \in \Sigma$

2. If R_1 and R_2 are regular expressions, then so are $(R_1 \cup R_2)$, $(R_1 \circ R_2)$, and (R_1^*)

Examples: (over $\Sigma = \{a, b, c\}$)

$(a \circ b)$ $((((a \circ (b^*)) \circ c) \cup (((a^*) \circ b))^*))$ $(\emptyset^*)$

Regular Expressions – Semantics

$L(R)$ = the language a regular expression describes

1. $L(\emptyset) = \emptyset$
2. $L(\varepsilon) = \{\varepsilon\}$
3. $L(a) = \{a\}$ for every $a \in \Sigma$
4. $L((R_1 \cup R_2)) = L(R_1) \cup L(R_2)$
5. $L((R_1 \circ R_2)) = L(R_1) \circ L(R_2)$
6. $L\left((R_1^*)\right) = (L(R_1))^*$

Regular Expressions – Example

$L(((a^*) \circ (b^*))) =$

a) $\{a^n b^n \mid n \geq 0\}$

b) $\{a^m b^n \mid m, n \geq 0\}$

c) $\{(ab)^n \mid n \geq 0\}$

d) $\{a, b\}^*$

1) $L(a) = \{a\}$

   $L(b) = \{b\}$

2) $L(a^*) = \{a^m \mid m \geq 0\}$

   $L(b^*) = \{b^n \mid n \geq 0\}$

3) $L((a^*) \circ (b^*)) =$

   $L((a^*)) \circ L((b^*)) =$

   $\{a^m b^n \mid m \geq 0, n \geq 0\}$

Simplifying Notation

Automata Tutor:

| means ∪

juxtaposition = ∘

- Omit ∘ symbol: $(ab) = (a \circ b)$

- Omit many parentheses, since union and concatenation are associative:

$$(a \cup b \cup c) = (a \cup (b \cup c)) = ((a \cup b) \cup c)$$

- Order of operations: Evaluate star, then concatenation, then union

$$ab^* \cup c = (a(b^*)) \cup c$$

Examples

Let $\Sigma = \{0, 1\}$

1. $\{w \mid w \text{ contains exactly one } 1\}$

   Try 1: $L(1) = \{1\}$

   Try 2: $= L(0^* 1 0^*)$

   $= \{0^m 1 0^n \mid m,n \geq 0\}$

2. $\{w \mid w \text{ has length at least 3 and its third symbol is } 0\}$

   Idea: $a\, a\, 0\, a^*$ $\rightsquigarrow$ $L((0 \cup 1)(0 \cup 1)\, 0\, (0 \cup 1)^*)$

   $\Sigma \Sigma 0 \Sigma^*$

3. $\{w \mid \text{every odd position of } w \text{ is } 1\}$

   $L((1(0 \cup 1))^* 1^*)$

   10
   11
   1010
   1011

   101

   also: $L((1(0 \cup 1))^* \cup (1(0 \cup 1))^* 1)$

Syntactic Sugar

- For alphabet Σ, the regex Σ represents $L(\Sigma) = \Sigma$

$\Sigma = \{a, b, c, \ldots, z\}$

means $(a \cup b \cup c \cup \ldots \cup z)$

- For regex R, the regex $R^+ = RR^*$

$L(R^+) = \{a_1 a_2 \ldots a_n \mid n \geq 1,$

and $a_i \in L(R)\ \forall i \}$

Regexes in the Real World

`grep` = globally search for a regular expression and print matching lines

```
$ grep '^xy*z' myfile
xyz
xyzde
xzz
xz
xyyz
xyyyz
xyyyyz
$ grep '^x.*z' myfile
xyz
xyzde
xxz
xzz
x\z
x*z
xz
x z
xYz
xyyz
xyyyz
xyyyyz
$ grep '^x\*z' myfile
x*z
$ grep '\\' myfile
x\z
$ 
```

Equivalence of Regular Expressions, NFAs, and DFAs

Regular Expressions Describe Regular Languages

Theorem: A language A is regular if and only if it is described by a regular expression

Today:

Theorem 1: Every regular expression has an equivalent NFA

Tuesday:

Theorem 2: Every NFA has an equivalent regular expression

Regular expression -> NFA

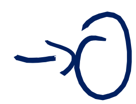
Theorem 1: Every regex has an equivalent NFA

Proof: Induction on size of a regex

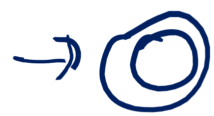
Language | NFA

Base cases:

$R = \emptyset$ $L(R) = \phi$



$R = \varepsilon$ $L(R) = \{\epsilon\}$



$R = a$ $L(R) = \{a\}$

Regular expression -> NFA

Theorem 1: Every regex has an equivalent NFA

Proof: Induction on size of a regex

Assuming IH, show every regex of size k+1 has an equiv. NFA

What should the inductive hypothesis be?

a) Suppose **some** regular expression of length k can be converted to an NFA

b) Suppose **every** regular expression of length k can be converted to an NFA

c) Suppose **every** regular expression of length **at most** k can be converted to an NFA

d) None of the above

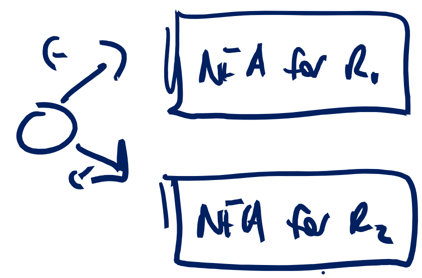
Regular expression -> NFA

Theorem 1: Every regex has an equivalent NFA

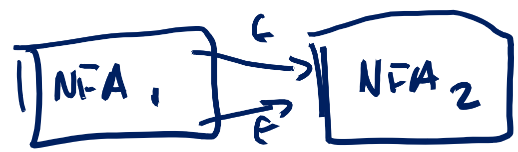
Proof: Induction on size of a regex

Inductive step:

$$R = (R_1 \cup R_2)$$



$$R = (R_1 R_2)$$



$$R = (R_1^*)$$