BU CS 332 – Theory of Computation

https://forms.gle/XT3v76KCagDQBsQL6

Lecture 6:

- Regexes = NFAs
- Non-regular languages

Reading:

Sipser Ch 1.3

"Myhill-Nerode" note

Mark Bun

September 21, 2021

- HW 2 due tonight
- Islam's OH moved to 3:30 - 4:30 MCS B33 (today only)

Regular Expressions – Syntax

A regular expression R is defined recursively using the following rules:

1. ε, $\emptyset$, and a are regular expressions for every $a \in \Sigma$

2. If R_1 and R_2 are regular expressions, then so are
   $(R_1 \cup R_2)$, $(R_1 \circ R_2)$, and (R_1^*)

Examples: (over $\Sigma = \{a, b, c\}$) (with simplified notation)

ab $\quad$ $ab^*c \cup (a^*b)^*$ $\quad$ $\emptyset$

Regular Expressions – Semantics

$L(R)$ = the language a regular expression describes

1. $L(\emptyset) = \emptyset$
2. $L(\varepsilon) = \{\varepsilon\}$
3. $L(a) = \{a\}$ for every $a \in \Sigma$
4. $L((R_1 \cup R_2)) = L(R_1) \cup L(R_2)$
5. $L((R_1 \circ R_2)) = L(R_1) \circ L(R_2)$
6. $L\left((R_1^*)\right) = (L(R_1))^*$

Example: $L(a^* b^*) = \{a^m b^n \mid m, n \geq 0\}$

Regular Expressions Describe Regular Languages

Theorem: A language A is regular if and only if it is described by a regular expression

Theorem 1: Every regular expression has an equivalent NFA

Theorem 2: Every NFA has an equivalent regular expression

Regular expression -> NFA

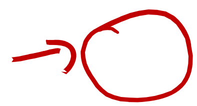
Theorem 1: Every regex has an equivalent NFA

Proof: Induction on size of a regex

$L(N)$

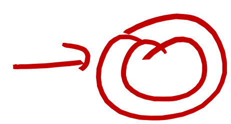
Base cases:

$R = \emptyset$



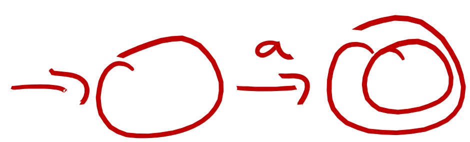
ϕ

$R = \varepsilon$



$\{\varepsilon\}$

$R = a$



$\{a\}$

Regular expression -> NFA

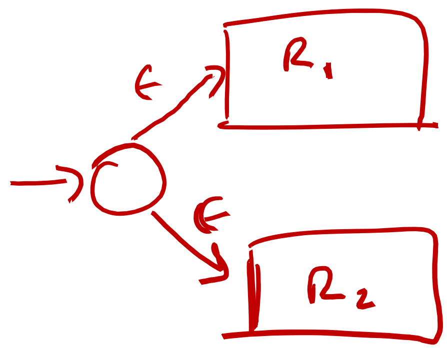
Theorem 1: Every regex has an equivalent NFA

Proof: Induction on size of a regex

$L(N)$

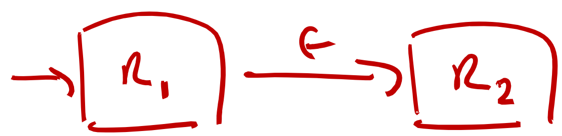
Inductive step:

$$R = (R_1 \cup R_2)$$



$L(R_1) \cup L(R_2)$

$$R = (R_1 R_2)$$



$L(R_1) \circ L(R_2)$

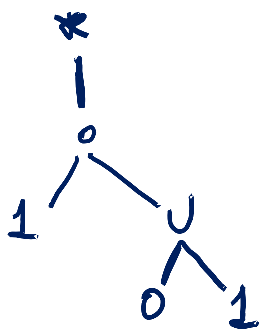
$$R = (R_1^*)$$

$(L(R_1))^*$

Example

Convert $(1(0 \cup 1))^*$ to an NFA

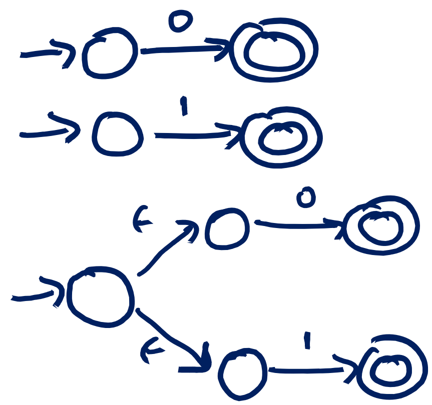
even length strings s.t. every odd position is 1



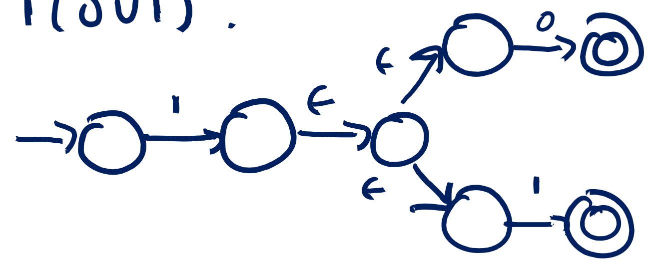
0 :

1 :

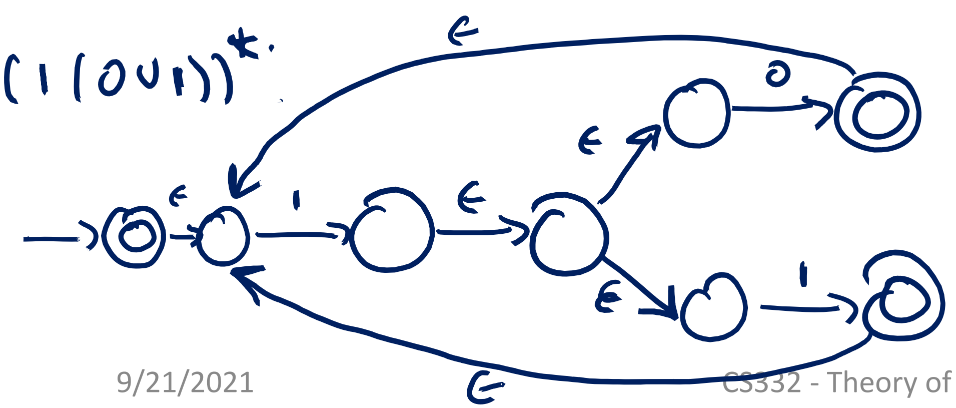
$0 \cup 1$:



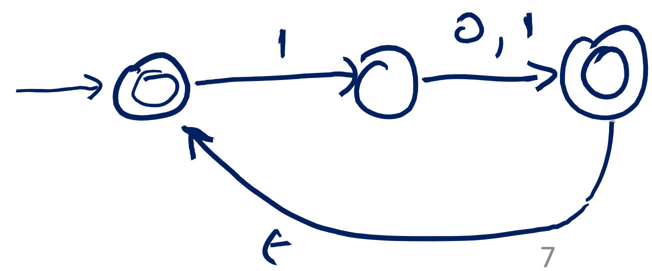
$1(0 \cup 1)$:



$(1(0 \cup 1))^*$:



Simplification:



Regular Expressions Describe Regular Languages

Theorem: A language A is regular if and only if it is described by a regular expression

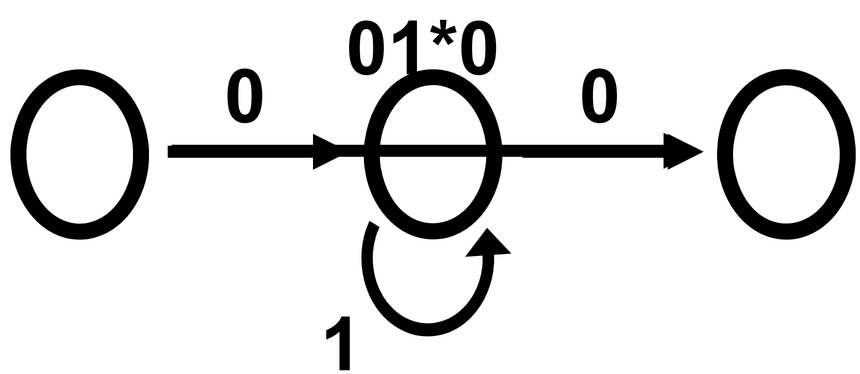
Theorem 1: Every regular expression has an equivalent NFA

Theorem 2: Every NFA has an equivalent regular expression

NFA -> Regular expression

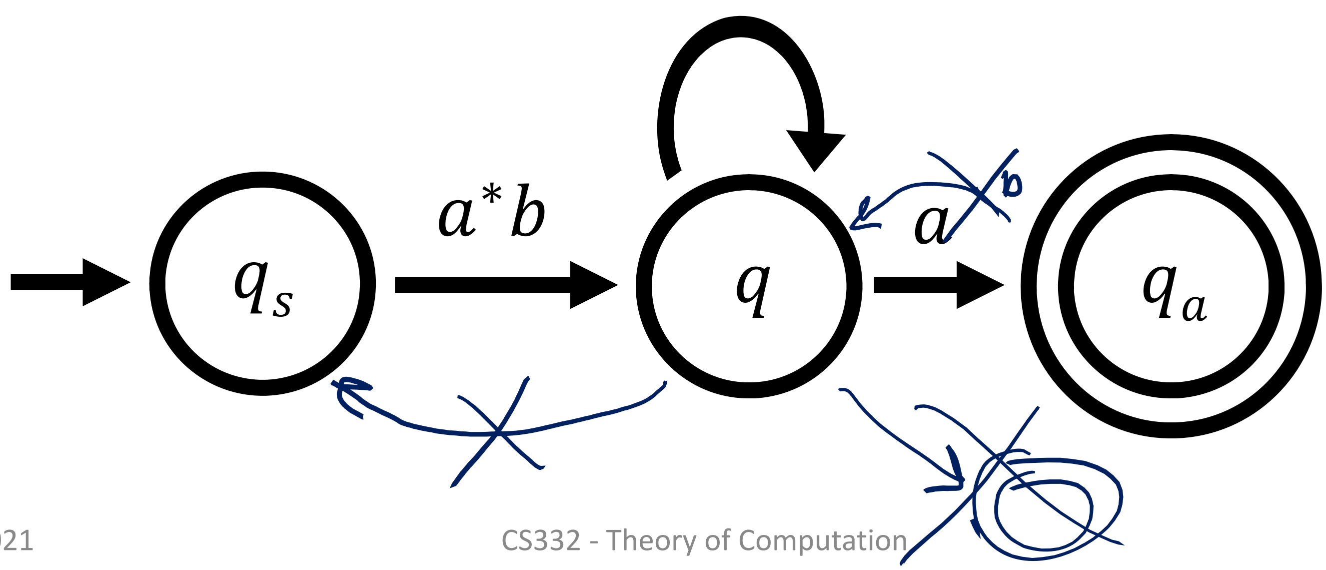
Theorem 2: Every NFA has an equivalent regex

Proof idea: Simplify NFA by “ripping out” states one at a time and replacing with regexes

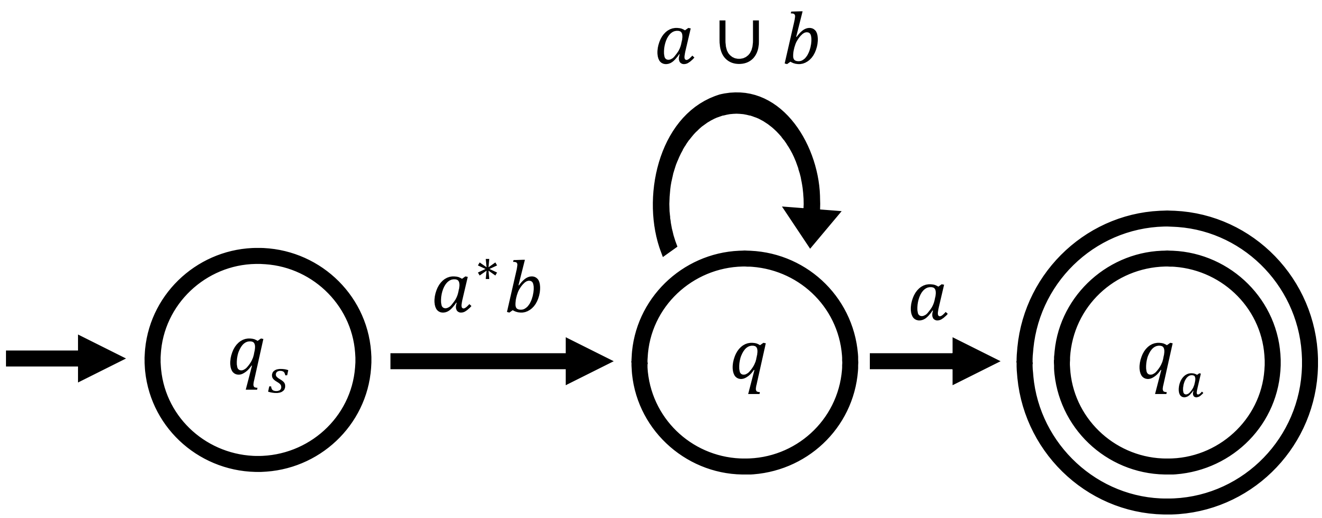


Generalized NFAs (GNFA)

- **Every transition is labeled by a regex**
- One start state with only outgoing transitions
- Only one accept state with only incoming transitions
- Start state and accept state are distinct



Generalized NFA Example



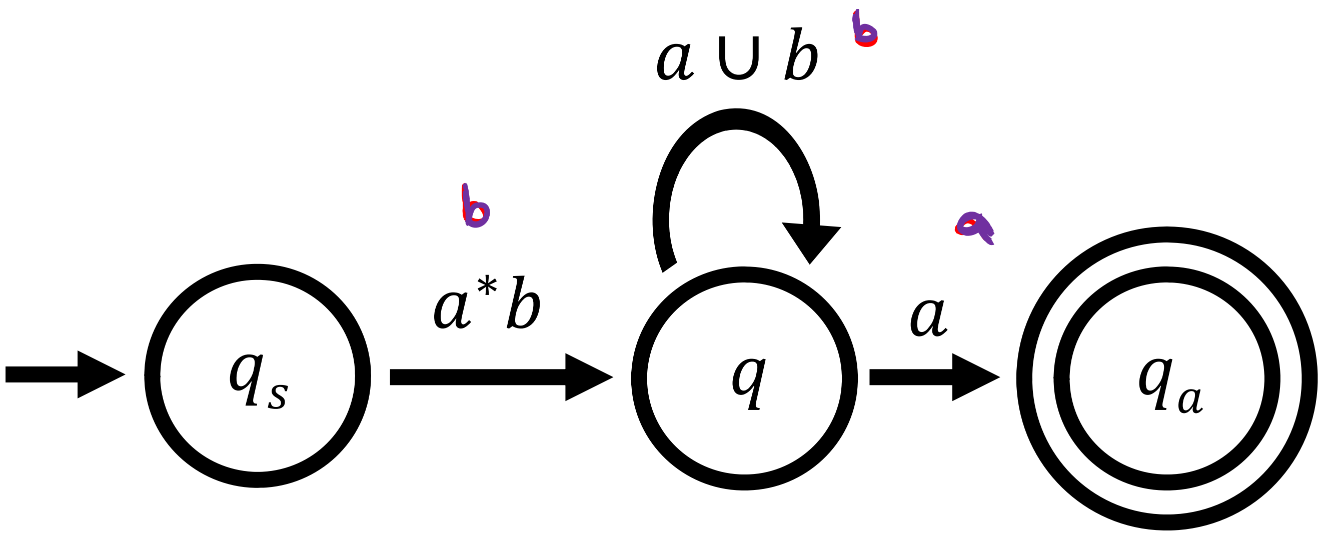
$$R(q_s, q) = a^* b$$

$$R(q_a, q) = \emptyset$$

$$R(q, q_s) = \emptyset$$

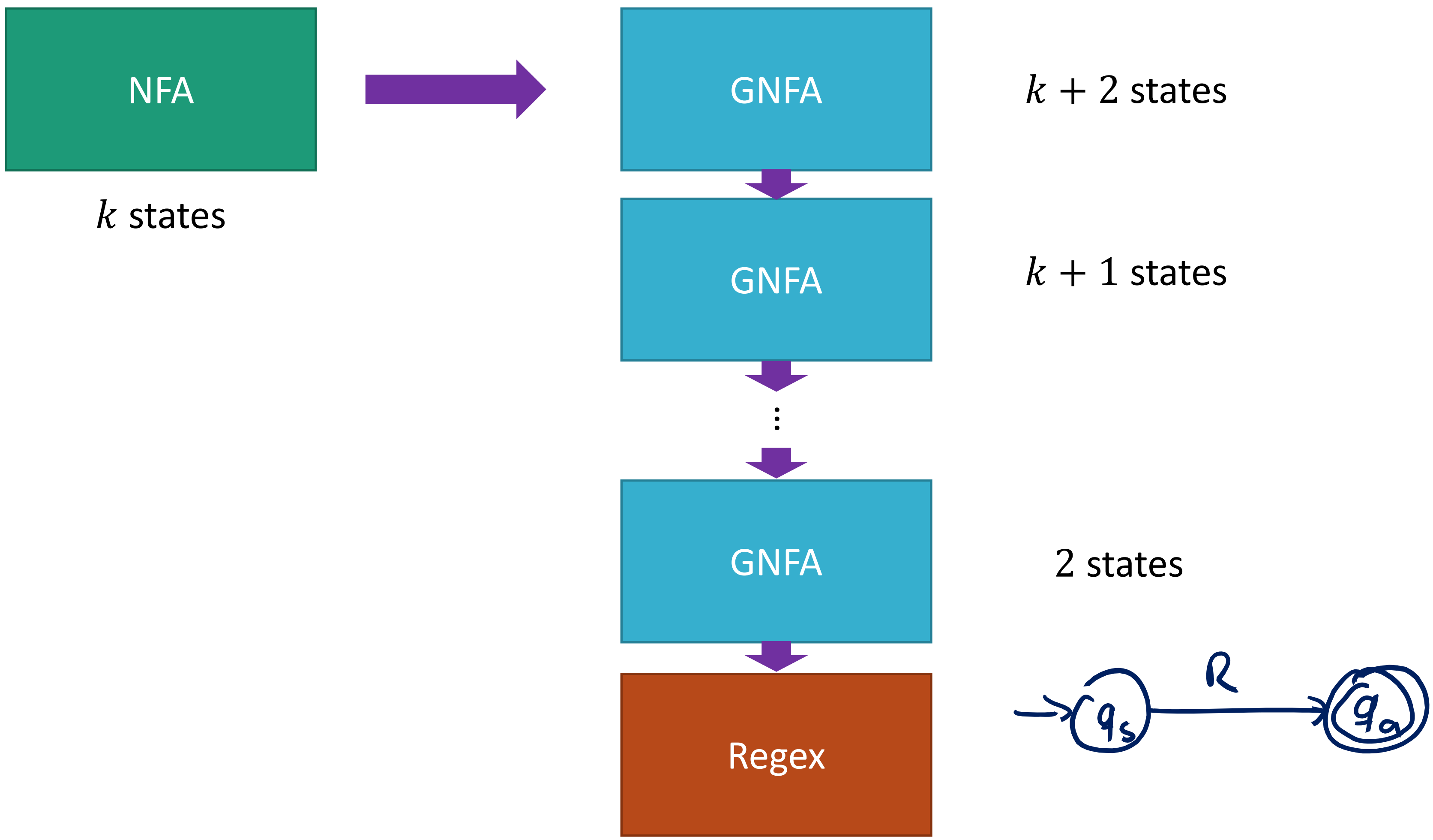
Which of these strings is accepted?

Which of the following strings is accepted by this GNFA?



a) ~~aaa~~
b) ~~$aabb$~~
c) ~~bbb~~
d) bba ✓

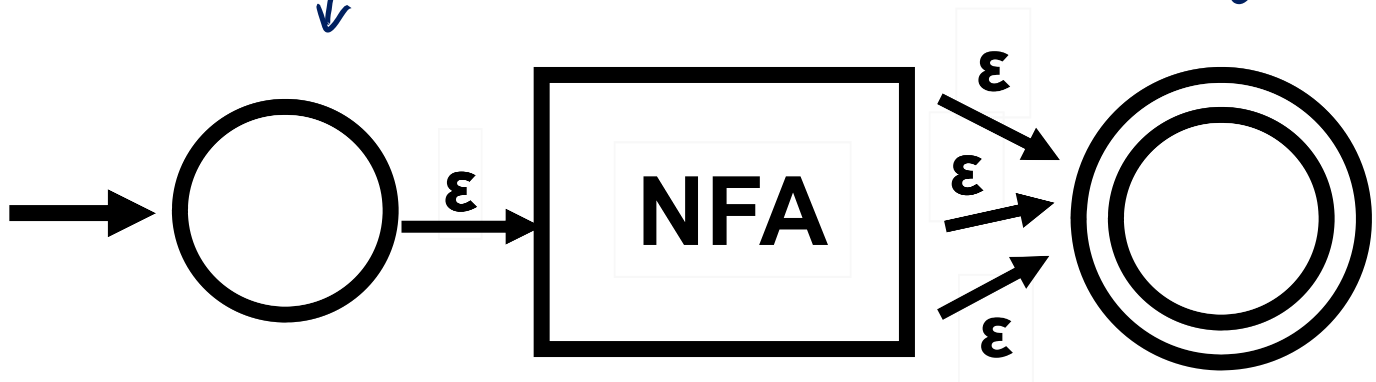
NFA -> Regular expression



NFA -> GNFA

start only has out arrows

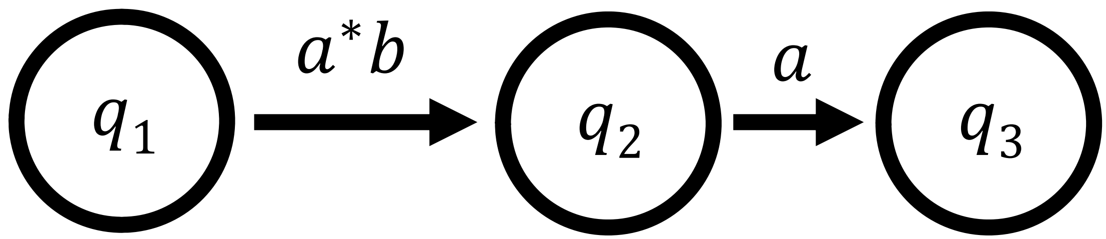
only one accept w/ only in arrows



- Add a new start state with no incoming arrows.
- Make a unique accept state with no outgoing arrows.

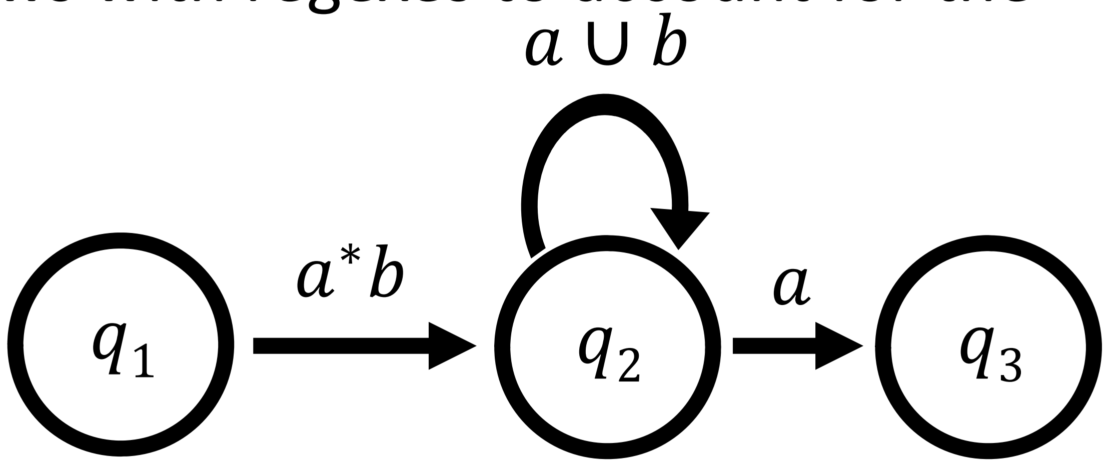
GNFA -> Regular expression

Idea: While the machine has more than 2 states, rip one out and relabel the arrows with regexes to account for the missing state

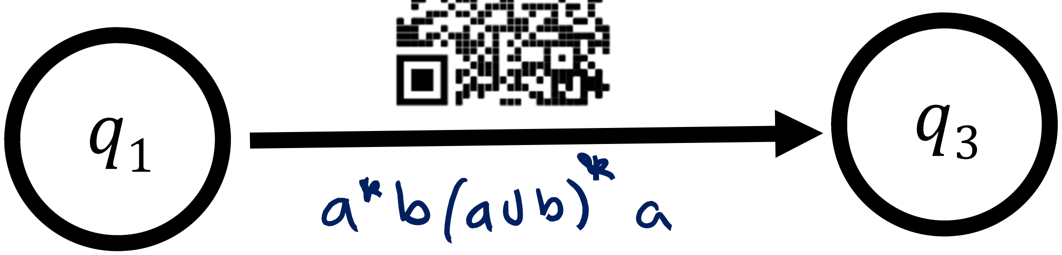


GNFA -> Regular expression

Idea: While the machine has more than 2 states, rip one out and relabel the arrows with regexes to account for the missing state



a) $a^*b(a \cup b)a$
b) $a^*b(a \cup b)^*a$
c) $a^*b \cup (a \cup b) \cup a$
d) None of the above



GNFA -> Regular expression

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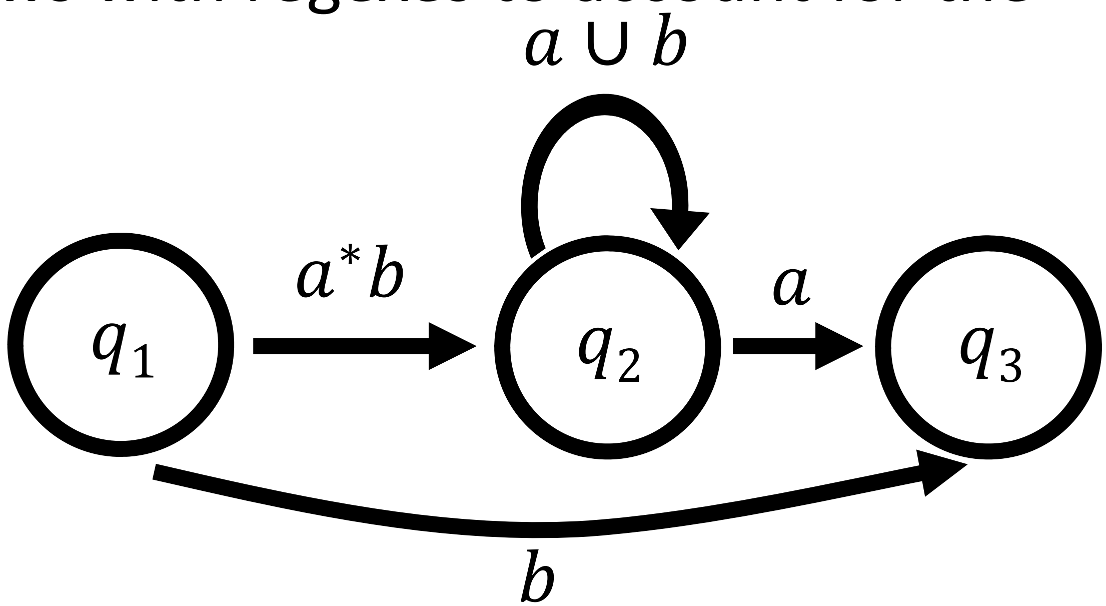
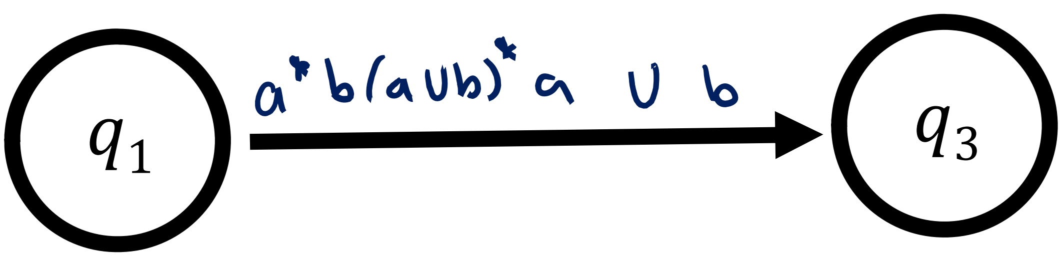
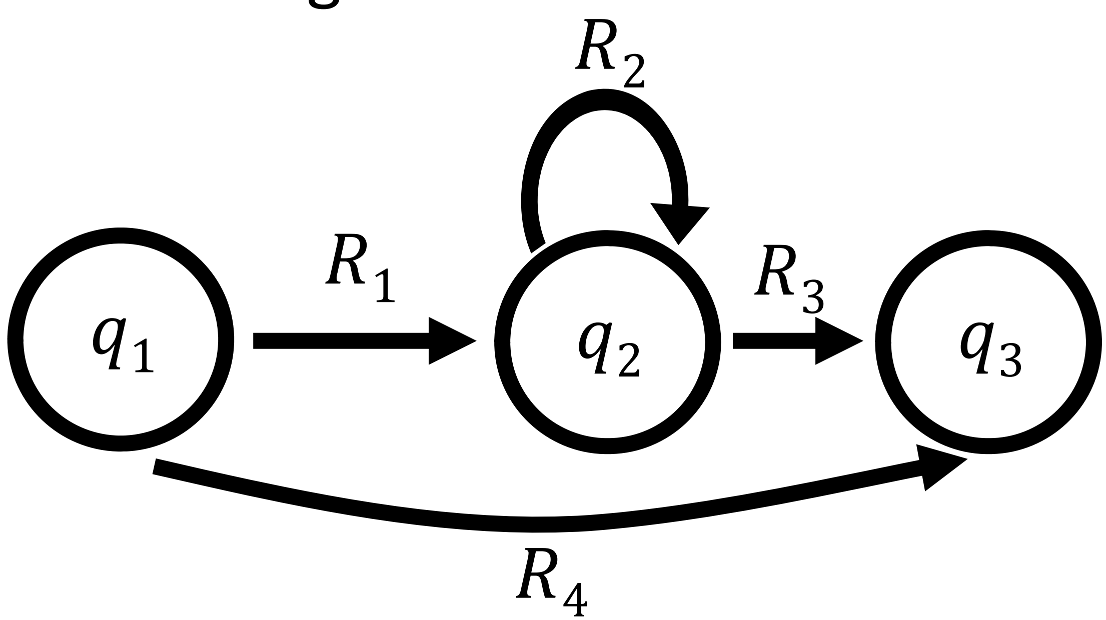
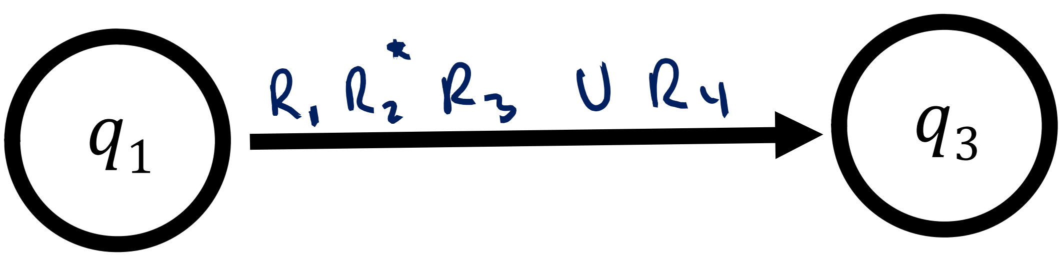


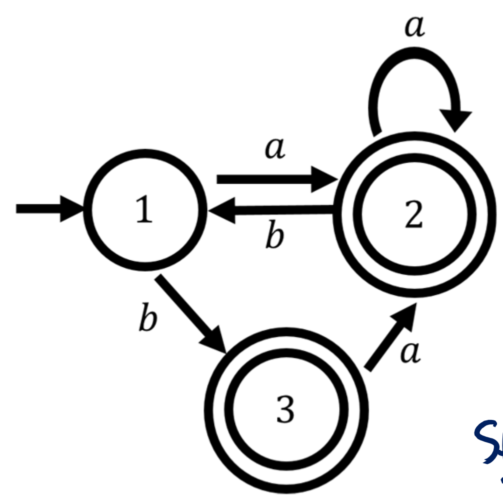


GNFA -> Regular expression

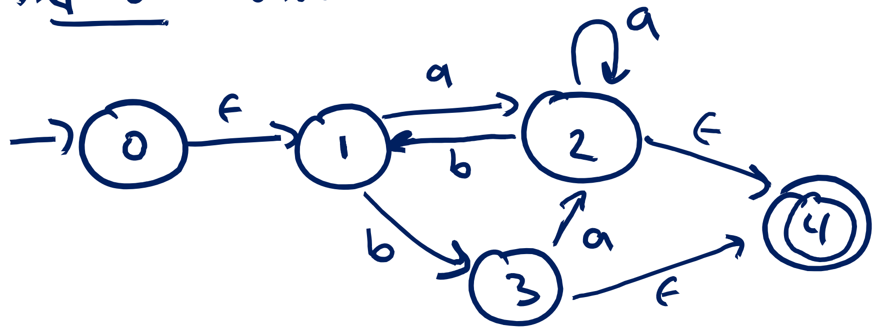
Idea: While the machine has more than 2 states, rip one out and relabel the arrows with regexes to account for the missing state







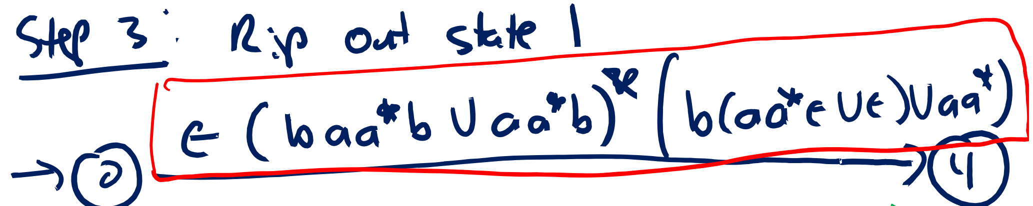
Step 0: Convert to GNFA



Step 1: Rip out state 2

Step 2: Rip out state 3

Step 3: Rip out state 1



$= (baa^*b \cup aa^*b)^*(ba^* \cup aa^*)$

$= (ba^+b \cup a^+b)^*(ba^* \cup a^+)$

Non-Regular Languages

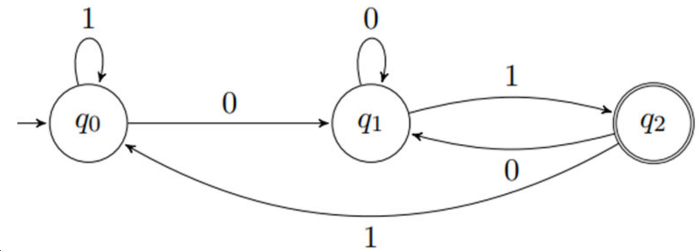
Motivating Questions

- We've seen techniques for showing that languages are regular
  - construct DFA
  - construct NFA
  - construct regex

- How can we tell if we've found the smallest DFA recognizing a language?
- Are all languages regular? How can we prove that a language is not regular?

An Example

Smallest possible →



$A = \{ w \in \{0,1\}^* \mid w \text{ ends with } 01 \}$

Claim: *Every* DFA recognizing A needs at least 3 states

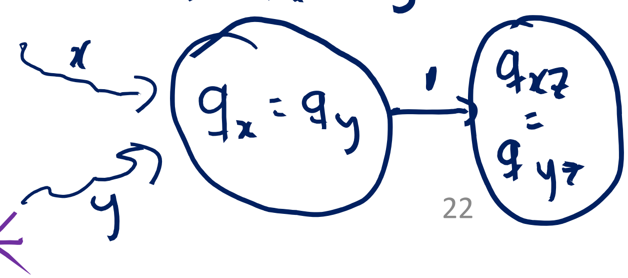
Proof: Let M be any DFA recognizing A. Consider running M on each of $x = \varepsilon, y = 0, w = 01$

Let q_x = state M reaches when reading x
q_y = " reading y
q_w = " reading w

Claim: q_x, q_y, q_w are all distinct

$q_w \neq q_x$, $q_w \neq q_y$ because q_w is an accept, q_x & q_y reject

$q_x \neq q_y$: Assume for contrad. that $q_x = q_y$.
Let $z = 1$. Then what does M do on
$xz = 1$, should reject and $yz = 01$? should accept



A General Technique

$A = \{ w \in \{0,1\}^* \mid w \text{ ends with } 01 \}$

z "is a distinguishing extension" for x and y

Definition: Strings x and y are **distinguishable** by L if there exists a string z such that exactly one of xz or yz is in L.

Ex. $x = \varepsilon, \quad y = 0$ $\quad z = 1 \quad xz \notin A \quad yz \in A$

Definition: A set of strings S is **pairwise distinguishable** by L if every pair of distinct strings $x, y \in S$ is distinguishable by L.

Ex. $S = \{\varepsilon, 0, 01\}$

$x = \epsilon,\ y = 0 :\ z = 1$

$x = \epsilon,\ y = 01 :\ z = \epsilon$

$x = 0,\ y = 01 :\ z = \epsilon$

A General Technique

Theorem: If S is pairwise distinguishable by L, then every DFA recognizing L needs at least $|S|$ states