BU CS 332 – Theory of Computation

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Lecture 12:

- Decidable Languages
- Universal TM

Reading:

Sipser Ch 4.1

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Last Time

Nondeterministic TMs

An NTM N accepts input w if when run on w it accepts on at least one computational branch

Church-Turing Thesis

v1: The basic TM (and all equivalent models) capture our intuitive notion of algorithms

v2: Any physically realizable model of computation can be simulated by the basic TM

Decidable Languages

1928 – The *Entscheidungsproblem*

The "Decision Problem"

TM

"mathematical statement"

Is there an algorithm which takes as input a formula (in first-order logic) and decides whether it's logically valid?

is the statement true or false?

Question: Can computers automate mathematicians?

Question: Can we automate theorems about regular languages?

Questions about regular languages

- Given a DFA D and a string w, does D accept input w?
- Given a DFA D, does D recognize the empty language?
- Given DFAs D_1, D_2, do they recognize the same language?

(Same questions apply to NFAs, regexes)

Goal: Formulate each of these questions as a language, and decide them using Turing machines

Questions about regular languages

Design a TM which takes as input a DFA D and a string w, and determines whether D accepts w

How should the input to this TM be represented?

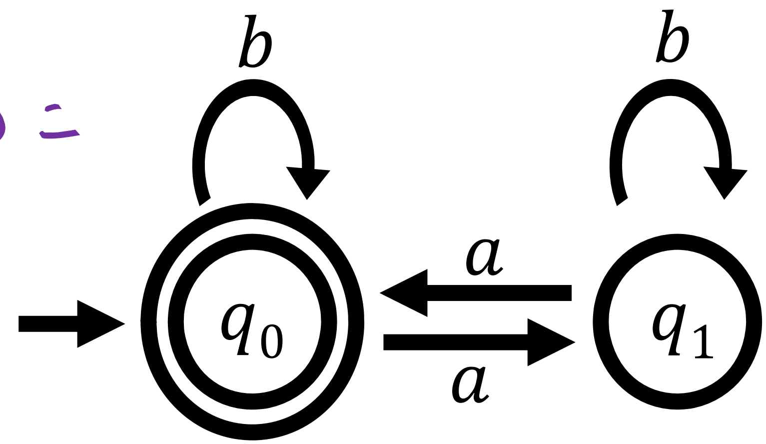
Let $D = (Q, \Sigma, \delta, q_0, F)$. List each component of the tuple separated by #

- Represent Q by ,-separated binary strings
- Represent Σ by ,-separated binary strings
- Represent $\delta : Q \times \Sigma \rightarrow Q$ by a ,-separated list of triples (p, a, q), …

Denote the encoding of D, w by $\langle D, w \rangle$

Example

$D =$



$Q = \{q_0, q_1\}$

$\Sigma = \{a, b\}$ ($a \to 0$, $b \to 1$)

$\delta(q_0, a) = q_1$, $\delta(q_0, b) = q_0$, ...

start q_0

$F = \{q_1\}$

$\langle D \rangle$: $\underbrace{0, 1}_{Q}$ # $\underbrace{0, 1}_{\Sigma}$ # $\underbrace{(0,0,1), (0,1,0), (1,0,0), (1,1,1)}_{\delta}$

$\underbrace{0}_{q_0}$ # $\underbrace{1}_{F}$

Alphabet of encoded DFA:

$\{0, 1, \#, ,, (,)\}$

Representation independence

Computability (i.e., decidability and recognizability) is **not** affected by the precise choice of encoding

M decides language about DFAs under encoding ⟨·⟩

Let [·] be a different encoding

Why? A TM can always convert between different (reasonable) encodings

N decides language under encoding [·]:

On input [D]:

1) Convert [D] to ⟨D⟩

2) Run M on input ⟨D⟩, accept if it accepts, reject otherwise.

From now on, we'll take ⟨ ⟩ to mean "any reasonable encoding"

A "universal" algorithm for recognizing regular languages

Computational problem:
Given DFA D, string w
Does D accept w?

$A_{\text{DFA}} = \{\langle D, w\rangle \mid \text{DFA } D \text{ accepts } w\}$

Theorem: A_{DFA} is decidable

Tape 2: $\boxed{w_1 \mid w_2 \mid \ldots \mid \dot{w}_i \mid \ldots \mid w_n}$ Tape 3: $\boxed{q \mid \delta \mid \ldots}$

Proof: Define a (high-level) 3-tape TM M on input $\langle D, w\rangle$:

1. Check if $\langle D, w\rangle$ is a valid encoding (reject if not)
2. Simulate D on w, i.e.,
   - Tape 2: Maintain w and head location of D
   - Tape 3: Maintain state of D, update according to δ

Omit if you want; understood to be a part of any algorithm

3. Accept if D ends in an accept state, reject otherwise

Analysis: 1) $\langle D, w\rangle \in A_{DFA} \Rightarrow D$ ends in accept on input $w \Rightarrow M$ accepts

2) $\langle D, w\rangle \notin A_{DFA} \Rightarrow$ either improperly formatted, or D ends in reject $\Rightarrow M$ rejects

Other decidable languages

$A_{\text{DFA}} = \{\langle D, w\rangle \mid \text{DFA } D \text{ accepts } w\}$

$A_{\text{NFA}} = \{\langle N, w\rangle \mid \text{NFA } N \text{ accepts } w\}$

$A_{\text{REX}} = \{\langle R, w\rangle \mid \text{regular expression } R \text{ generates } w\}$

NFA Acceptance

Your TM should: Accept $\langle N, w\rangle$ if N accepts w
Reject $\langle N, w\rangle$ if N does not accept w

Which of the following describes a **decider** for $A_{\text{NFA}} = \{\langle N, w\rangle \mid \text{NFA } N \text{ accepts } w\}$?

a) Using a deterministic TM, simulate N on w, always making the first nondeterministic choice at each step. Accept if it accepts, and reject otherwise.

b) Using a deterministic TM, simulate all possible choices of N on w for 1 step of computation, 2 steps of computation, etc. Accept whenever some simulation accepts.

N:  w: 0

c) Use the subset construction to convert N to an equivalent DFA M. Simulate M on w, accept if it accepts, and reject otherwise.

Regular Languages are Decidable

Theorem: Every regular language L is decidable

Proof 1: If L is regular, it is recognized by a DFA D. Convert this DFA to a TM M. Then M decides L.

Proof 2: If L is regular, it is recognized by a DFA D. The following TM M_D decides L.

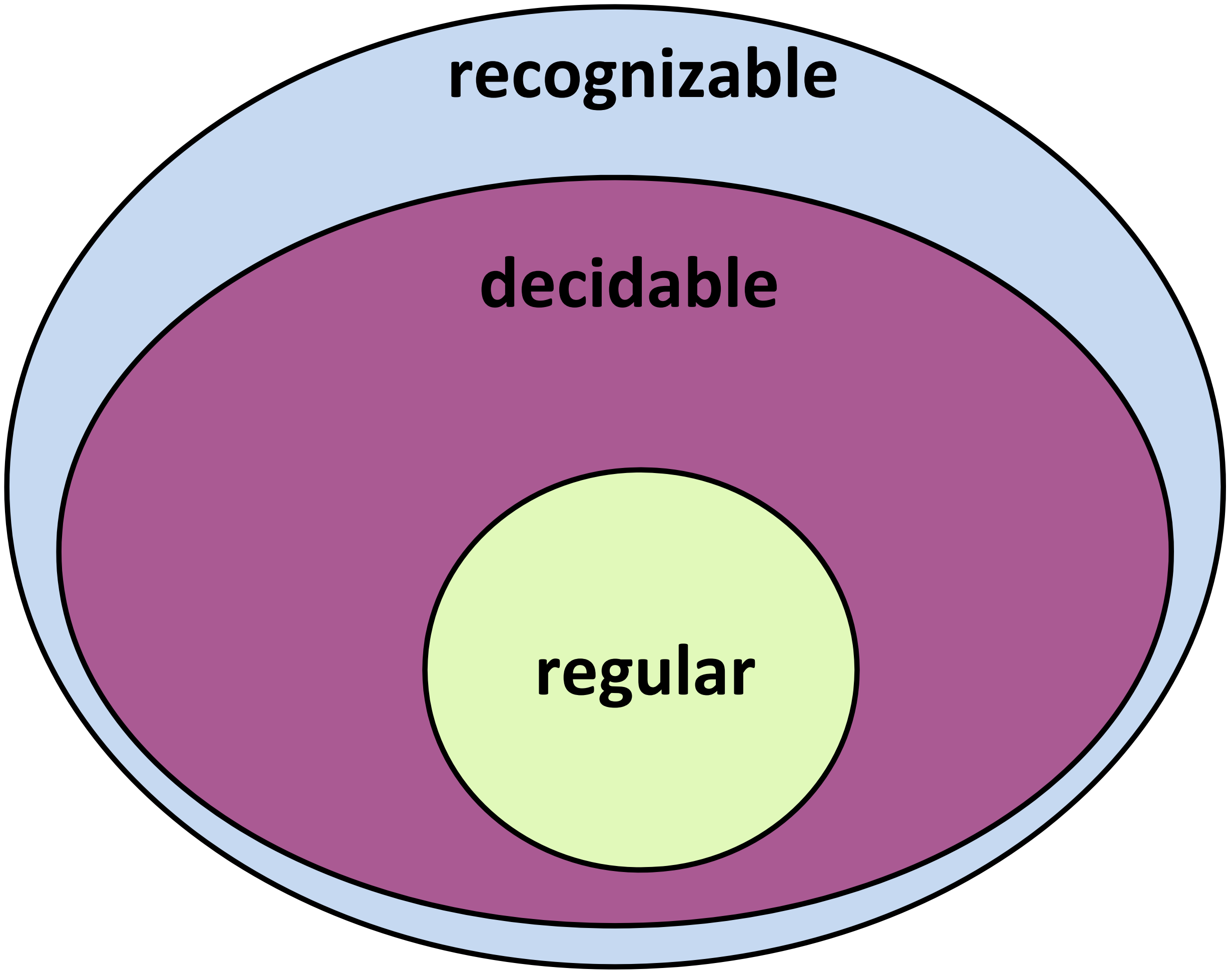
On input w:

1. Run the decider for A_{DFA} on input $\langle D, w \rangle$
2. Accept if the decider accepts; reject otherwise

Analysis: 1) If $w \in L$, then D accepts $w \Rightarrow \langle D, w \rangle \in A_{DFA} \Rightarrow M_D$ accepts

2) If $w \notin L$, then D does not accept $w \Rightarrow \langle D, w \rangle \notin A_{DFA} \Rightarrow M_D$ rejects

Classes of Languages



More Decidable Languages: Emptiness Testing

Theorem: $E_{\text{DFA}} = \{\langle D \rangle \mid D \text{ is a DFA such that } L(D) = \emptyset\}$ is decidable

Given DFA D, is it the case that D accepts no strings?

Proof: The following TM decides E_{DFA}

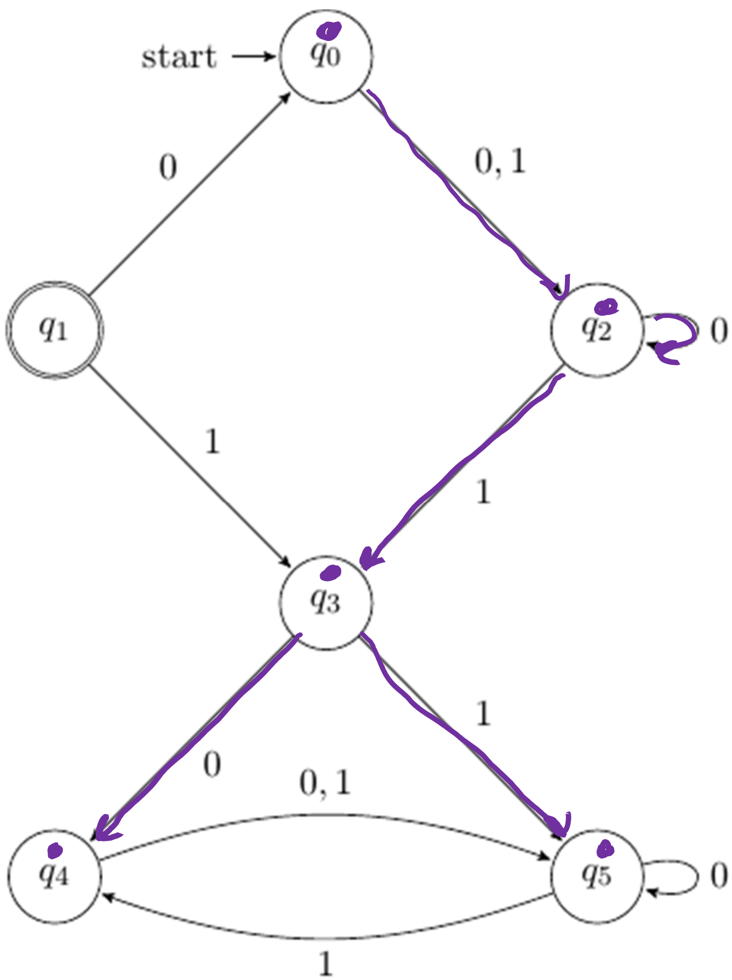
- If $L(D) = \emptyset$, there are no reachable accept states => TM accepts
- If $L(D) \neq \emptyset$, $\exists$ a reachable accept state => TM rejects

On input $\langle D \rangle$, where D is a DFA with k states:

1. Perform k steps of breadth-first search on state diagram of D to determine if an accept state is reachable from the start state
2. Reject if a DFA accept state is reachable; accept otherwise

E_{DFA} Example

D =



BFS traversal:

q_0

q_2

q_3

q_4, q_5

No accept states are reachable

$\Rightarrow L(D) = \phi$

reject input <D>.

New Deciders from Old: Equality Testing

$EQ_{\text{DFA}} = \{\langle D_1, D_2\rangle \mid D_1, D_2 \text{ are DFAs and } L(D_1) = L(D_2)\}$

Theorem: EQ_{DFA} is decidable

$A \triangle B = \{(w \in A \text{ and } w \notin B) \text{ or } (w \in B \text{ and } w \notin A)\}$

Proof: The following TM decides EQ_{DFA}

On input $\langle D_1, D_2\rangle$, where $\langle D_1, D_2\rangle$ are DFAs:

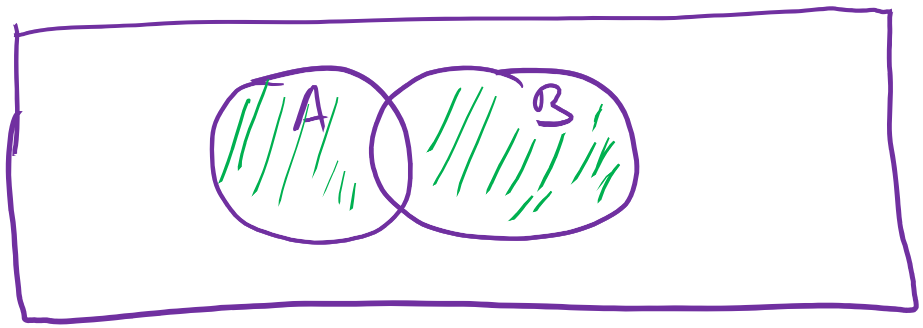
1. Construct DFA D recognizing the **symmetric difference** $L(D_1) \triangle L(D_2)$ $= \{ w \mid$ exactly one of D_1, D_2 accept $w\}$
2. Run the decider for E_{DFA} on $\langle D\rangle$ and return its output

Analysis:

1) If $L(D_1) = L(D_2)$, then $L(D_1) \triangle L(D_2) = \emptyset \Rightarrow$ decider for E_{DFA} accepts $\Rightarrow$ TM accepts
2) If $L(D_1) \neq L(D_2) \Rightarrow L(D_1) \triangle L(D_2) \neq \emptyset \Rightarrow$ decider for E_{DFA} rejects $\Rightarrow$ TM rejects

Symmetric Difference

$$A \triangle B = \{w \mid w \in A \text{ or } w \in B \text{ but not both}\}$$



$$A \triangle B = (A \setminus B) \cup (B \setminus A)$$

$$= (A \cap \overline{B}) \cup (B \cap \overline{A})$$

If A, B recognized by DFAs, can use complement/intersection/union constructions + subset construction to get DFA for $A \triangle B$.

Universal Turing Machine

Meta-Computational Languages

$A_{\text{DFA}} = \{\langle D, w\rangle \mid \text{DFA } D \text{ accepts } w\}$

$A_{\text{TM}} = \{\langle M, w\rangle \mid \text{TM } M \text{ accepts } w\}$

$E_{\text{DFA}} = \{\langle D\rangle \mid \text{DFA } D \text{ recognizes the empty language } \emptyset\}$

$E_{\text{TM}} = \{\langle M\rangle \mid \text{TM } M \text{ recognizes the empty language } \emptyset\}$

$EQ_{\text{DFA}} = \{\langle D_1, D_2\rangle \mid D_1 \text{ and } D_2 \text{ are DFAs}, L(D_1) = L(D_2)\}$

$EQ_{\text{TM}} = \{\langle M_1, M_2\rangle \mid M_1 \text{ and } M_2 \text{ are TMs}, L(M_1) = L(M_2)\}$

The Universal Turing Machine

$A_{\mathrm{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM that accepts input } w\}$

Theorem: A_{TM} is Turing-recognizable

Computational problem:
Given TM M and input w
does M accept w?

The following "Universal TM" U recognizes A_{TM}

On input $\langle M, w\rangle$:

1. Simulate running M on input w
2. If M accepts, accept. If M rejects, reject.

Analysis:

1) If $\langle M,w\rangle \in A_{TM} \Rightarrow$ M accepts w $\Rightarrow$ U accepts $\langle M,w\rangle$

2) If $\langle M,w\rangle \notin A_{TM} \Rightarrow$ either $\begin{cases} \text{M rejects w} \Rightarrow \text{U rejects } \langle M,w\rangle \\ \text{M loops on w} \Rightarrow \text{U loops on } \langle M,w\rangle \end{cases}$

Universal TM and A_{TM}

Why is the Universal TM not a decider for A_{TM}?

The following "Universal TM" U recognizes A_{TM}

On input $\langle M, w \rangle$:
1. Simulate running M on input w
2. If M accepts, accept. If M rejects, reject.

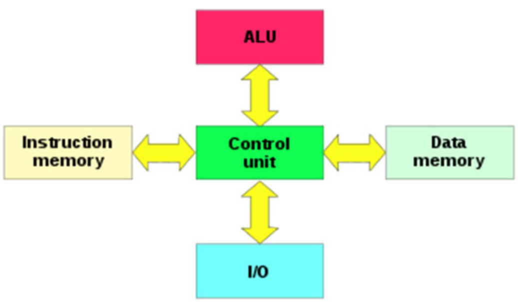
a) It may reject inputs $\langle M, w \rangle$ where M accepts w
b) It may accept inputs $\langle M, w \rangle$ where M rejects w
c) It may loop on inputs $\langle M, w \rangle$ where M loops on w
d) It may loop on inputs $\langle M, w \rangle$ where M accepts w

More on the Universal TM

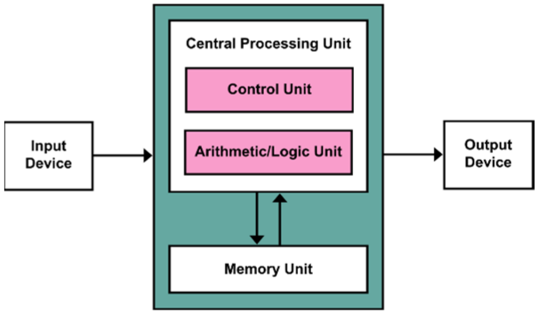
"It is possible to invent a single machine which can be used to compute any computable sequence. If this machine **U** is supplied with a tape on the beginning of which is written the S.D ["standard description"] of some computing machine **M**, then **U** will compute the same sequence as **M**."

- Turing, "On Computable Numbers…" 1936

- Foreshadowed general-purpose programmable computers
- No need for specialized hardware: Virtual machines as software



Harvard architecture:
Separate instruction and data pathways



von Neumann architecture:
Programs can be treated as data

Undecidability

A_{TM} is Turing-recognizable via the Universal TM

…but it turns out A_{TM} (and $E_{\mathrm{TM}}, EQ_{\mathrm{TM}}$) is **undecidable**

i.e., computers cannot solve these problems no matter how much time they are given