BU CS 332 – Theory of Computation

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Lecture 15:

- Review mid-semester feedback
- More on Reductions

Reading:

Sipser Ch 5.1

Mark Bun

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What helps you learn best?

- Discussion sections (17)
- In-class examples / walkthroughs (12)
- Lectures in general (10)
- Use of slides, annotations (8)
- Interaction in lecture, polls (6)
- Homework – useful, appropriate length/difficulty (6)
- Office hours (4)
- Course organization, perspective (2)
- Piazza use (2)
- Automata Tutor, TM simulator (1)
- Reading (1)

What hinders your learning?

- Automata Tutor / Morphett (1)
- Turing machines (1)

- Annotation readability (4)
- Not enough concrete examples in class (3)
- Identifying differences in definitions / types (1)
- Practice problems not exhaustive of material (1)
- Slides difficult to understand (2)
- Polls not useful (1)
- Hard to see or hear from back (2)
- Chalkboard use (2)
- Classroom distractions (1)
- Lectures boring (2)
- Classroom too warm (1)
- Lectue pace too fast (1)

- Can't make office hours (4)
- Environment not collaborative (1)

- Required discussions (1)
- Discussions in general (1)
- Discussion pace too slow (1)
- Lack of synchronization between discussion and lecture (1)

- Can't understand what HW problems are asking for (2)
- Proofs, proof assignments on homework (1)
- Homework too time-consuming, too difficult (3)
- Transferring lecture knowledge to homework (2)
- Grading (2)

Suggestions for course improvement

- More office hours (1)
- Zoom office hours (2)
- Don't require discussions / lecture attendance (1)
- Extend "late submission" deadline (1)
- Release grade statistics (1)
- Point to outside references (1)

- More examples (2)
- More polls, interaction (1)
- Slower lectures with more pauses (1)
- Introduce more material during lectures (1)
- More examples in class that are similar to homework (1)
- Review prerequisite material when needed (1)
- Clarify what parts of the material are most important (1)
- Record lectures (4)
- More programming examples (1)

- Use a mic (1)
- More in-class problem solving (1)
- Give more intuition leading into proofs before giving the proofs (1)
- More programming examples / exercises (2)
- More proof-based problem-solving examples (1)

- Fewer discussion problems / more time to discuss each (1)
- Synchronize discussion with previous lectures (1)
- More explanation of solutions during discussion (1)

- Shorter, but more difficult homework (1)
- Longer, but easier, homework (2)
- Make difficulty of lectures / homework closer (1)
- More homework hints (1)
- More practice problems (1)

Clarity of expectations

- Seems mostly clear
- Participation: Base grade determined by polls, discussion worksheets; other participation is "bonus"
- Reminder of resources to take advantage of:
  - Sipser textbook
  - Lectures (slides, ~~recordings~~)
  - Discussions (in-class meetings, posted slides)
  - Homework feedback, posted solutions
  - Office hours
  - Piazza

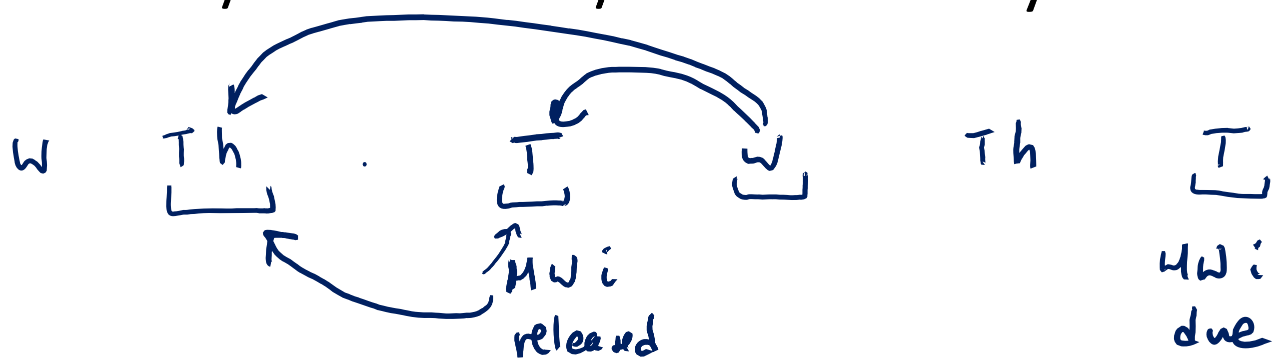
- See Lecture 1, Slides 13-17 for more advice

Suggestions for self-improvement

- Keep up with readings (17)
- Review lecture / discussion materials (7)
- Attend more office hours (7)
- Time management (6)
- Do example problems in Sipser (5)
- Participate in class more actively (2)
- More organized note-taking (1)

Proposed Course Modifications

- Poll for more office hours
- Synchronize lecture / discussion / homework cycle correctly



- Homework more approachable and useful
  - Gradient from easier (mechanical) to harder (creative) questions
  - Mechanical problems closer to discussion / lecture examples

Reductions

Reductions

A reduction from problem A to problem B is an algorithm for problem A which uses an algorithm for problem B as a subroutine

If such a reduction exists, we say "A reduces to B"

Positive uses: If A reduces to B and B is decidable, then A is also decidable

Ex. E_{DFA} is decidable $\Rightarrow$ EQ_{DFA} is decidable

Negative uses: If A reduces to B and A is undecidable, then B is also undecidable

Ex. A_{TM} is undecidable $\Rightarrow$ $HALT_{\mathrm{TM}}$ is decidable

Halting Problem

Computational problem: Given a program (TM) and input w, does that program halt (either accept or reject) on input w?

Formulation as a language:

$$HALT_{\text{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM that halts on input } w\}$$

Ex. M = "On input x (a natural number in binary):

For each $y = 1, 2, 3, \ldots$:

If $y^2 = x$, accept. Else, continue."

M' = "On input x (a natural number in binary):

For each $y = 1, 2, 3, \ldots, x$:

If $y^2 = x$, accept. Else, continue.

Reject."

Halting Problem

$$HALT_{\text{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM that halts on input } w\}$$

Theorem: $HALT_{\text{TM}}$ is undecidable

Proof: Suppose for contradiction that there exists a decider H for $HALT_{\text{TM}}$. We construct a decider for V for A_{TM} as follows:

On input $\langle M, w\rangle$: (input to A_{TM})
1. Run H on input $\langle M, w\rangle$
2. If H rejects, reject
3. If H accepts, run M on w
4. If M accepts, accept

   Otherwise, reject.

[Preprocess input to check if M will halt on w

[Simulate M on w

Claim: V decides A_{TM}

1) $\langle M,w\rangle \in A_{TM} \Rightarrow M$ accepts w
$\Rightarrow \langle M,w\rangle \in HALT_{TM}$
$\Rightarrow H$ accepts

Line 4: M accepts $w \Rightarrow V$ accepts

2) $\langle M,w\rangle \notin A_{TM} \Rightarrow M$ does not accept w

Either: a) M rejects $w \Rightarrow \langle M,w\rangle \in HALT_{TM}$
$\Rightarrow H$ accepts
$\Rightarrow V$ rejects

b) M loops on $w \Rightarrow \langle M,w\rangle \notin HALT$
$\Rightarrow H$ rejects
$\Rightarrow V$ rejects

This is a reduction from A_{TM} to $HALT_{\text{TM}}$

Halting Problem

Computational problem: Given a program (TM) and input w, does that program halt on input w?

- A central problem in formal verification
- Dealing with undecidability in practice:
  - Use heuristics that are correct on most real instances, but may be wrong or loop forever on others
  - Restrict to a "non-Turing-complete" subclass of programs for which halting is decidable
  - Use a programming language that lets a programmer specify hints (e.g., loop invariants) that can be compiled into a formal proof of halting

Emptiness testing for TMs

Computational Problem: Given a TM M, is the language recognized by M empty?

$$E_{\text{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$$

Theorem: E_{TM} is undecidable

Proof: Suppose for contradiction that there exists a decider R for E_{TM}. We construct a decider for A_{TM} as follows:

On input $\langle M, w \rangle$:

1. Run R on input ???

Run R on $\langle M \rangle$?

$$R(\langle M \rangle) = \begin{cases} \text{accept} & L(M) = \emptyset \\ \text{reject} & L(M) \neq \emptyset \end{cases}$$

Cannot distinguish between:

1) M accepts w
2) M does not accept w but M accepts something else

This is a reduction from A_{TM} to E_{TM}

Emptiness testing for TMs

Want: If $\langle M, w \rangle \in A_{TM}$, R reject
If $\langle M, w \rangle \notin A_{TM}$, R accept

$$E_{\text{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$$

Theorem: E_{TM} is undecidable

Proof: Suppose for contradiction that there exists a decider R for E_{TM}. We construct a decider for A_{TM} as follows:

On input $\langle M, w \rangle$:

1. Construct a TM N as follows:

$\langle M, w \rangle \in A_{TM} \iff R(\langle N \rangle)$ rejects
$\iff L(N)$ non-empty

2. Run R on input $\langle N \rangle$
3. If R rejects, accept. Otherwise, reject

What do we want out of machine N?

a) $L(N)$ is empty iff M accepts w
b) $L(N)$ is non-empty iff M accepts w
c) $L(M)$ is empty iff N accepts w
d) $L(M)$ is non-empty iff N accepts w

This is a reduction from A_{TM} to E_{TM}

Emptiness testing for TMs

$$E_{\text{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$$

Theorem: E_{TM} is undecidable

Proof: Suppose for contradiction that there exists a decider R for E_{TM}. We construct a decider for A_{TM} as follows:

On input $\langle M, w \rangle$:

1. Construct a TM N as follows:

   "On input x: Ignore x

   Run M on w and output the result."

2. Run R on input $\langle N \rangle$

3. If R rejects, accept. Otherwise, reject

Claim: $L(N) \neq \emptyset \iff M$ accepts w

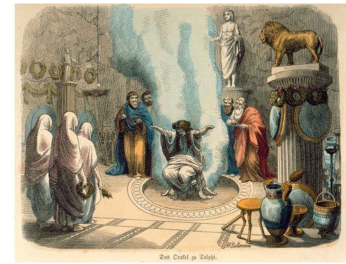
Proof:

1) M accepts w: $L(N) = \{x \mid N \text{ accepts } x\} = \Sigma^*$

2) M does not accept w: $L(N) = \emptyset$

This is a reduction from A_{TM} to E_{TM}

Interlude: Formalizing Reductions (Sipser 6.3)



Informally: A reduces to B if a decider for B can be used to construct a decider for A

One way to formalize:

- An *oracle* for language B is a device that can answer questions "Is $w \in B$?"
- An *oracle TM* M^B is a TM that can query an oracle for B in one computational step

A is Turing-reducible to B (written $A \leq_T B$) if there is an oracle TM M^B deciding A

Equality Testing for TMs

Don't be confused:
showed EQ_{DFA} decidable by showing
EQ_{DFA} reduces to E_{DFA}

$$EQ_{\text{TM}} = \{\langle M_1, M_2\rangle \mid M_1, M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$$

Theorem: EQ_{TM} is undecidable

Proof: Suppose for contradiction that there exists a decider R for EQ_{TM}. We construct a decider for E_{TM} as follows:

On input $\langle M\rangle$: Input to E_{TM}

1. Construct TMs N_1, N_2 as follows:

   N_1 = N_2 =

2. Run R on input $\langle N_1, N_2\rangle$
3. If R accepts, accept. Otherwise, reject.

This is a reduction from E_{TM} to EQ_{TM}

Equality Testing for TMs

$L(M) = \emptyset \Rightarrow L(N_2) = (L(N_1) = \emptyset)$
$L(M) \neq \emptyset \Rightarrow L(N_2) \neq L(N_1)$

What do we want out of the machines N_1, N_2?

a) $L(M) = \emptyset$ iff $N_1 = N_2$ b) $L(M) = \emptyset$ iff $L(N_1) = L(N_2)$

c) $L(M) = \emptyset$ iff $N_1 \neq N_2$ d) $L(M) = \emptyset$ iff $L(N_1) \neq L(N_2)$

On input $\langle M \rangle$:

1. Construct TMs N_1, N_2 as follows:

   N_1 = M

   N_2 = "On input x: // $L(N_2) = \emptyset$
   Reject"

2. Run R on input $\langle N_1, N_2 \rangle$

   $\Leftrightarrow L(N_1) = L(N_2)$
   R accepts $\Leftrightarrow \langle M \rangle \in E_{TM} \Leftrightarrow L(M) = \emptyset$

3. If R accepts, accept. Otherwise, reject.

This is a reduction from E_{TM} to EQ_{TM}

Equality Testing for TMs

$$EQ_{\text{TM}} = \{\langle M_1, M_2\rangle \mid M_1, M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$$

Theorem: EQ_{TM} is undecidable

Proof: Suppose for contradiction that there exists a decider R for EQ_{TM}. We construct a decider for A_{TM} as follows:

On input $\langle M\rangle$:

1. Construct TMs N_1, N_2 as follows:

   N_1 = N_2 =

2. Run R on input $\langle N_1, N_2\rangle$
3. If R accepts, accept. Otherwise, reject.

This is a reduction from E_{TM} to EQ_{TM}