Lecture 17:

• Mapping Reductions

Reading:
Sipser Ch 5.3

Mark Bun
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Take-home part of test 2 due Wednesday, 11:59 PM
Reductions

A reduction from problem $A$ to problem $B$ is an algorithm for problem $A$ which uses an algorithm for problem $B$ as a subroutine.

If such a reduction exists, we say “$A$ reduces to $B$”

Positive uses: If $A$ reduces to $B$ and $B$ is decidable, then $A$ is also decidable.

Ex. $E_{DFA}$ is decidable $\Rightarrow E_{Q_{DFA}}$ is decidable

Negative uses: If $A$ reduces to $B$ and $A$ is undecidable, then $B$ is also undecidable.

Ex. $E_{TM}$ is undecidable $\Rightarrow E_{Q_{TM}}$ is undecidable
What’s wrong with the following “proof”?

Bogus “Theorem”: $A_{TM}$ is not Turing-recognizable

Bogus “Proof”: Let $R$ be an alleged recognizer for $A_{TM}$. We construct a recognizer $S$ for unrecognizable language $\overline{A_{TM}}$:

On input $\langle M, w \rangle$:
1. Run $R$ on input $\langle M, w \rangle$
2. If $R$ accepts, reject. Otherwise, accept.

If $M$ loops on $w$, then $\langle M, w \rangle \in \overline{A_{TM}}$
But, $S(\langle M, w \rangle)$ loops forever, so behavior of $S$ is not correct.

This sure looks like a reduction from $\overline{A_{TM}}$ to $A_{TM}$
Mapping Reductions: Motivation

1. How do we formalize the notion of a reduction?
2. How do we use reductions to show that languages are unrecognizable?
3. How do we protect ourselves from accidentally “proving” bogus statements about recognizability?
 Computable Functions

Definition:

A function \( f: \Sigma^* \rightarrow \Sigma^* \) is **computable** if there is a TM \( M \) which, given as input any \( w \in \Sigma^* \), halts with only \( f(w) \) on its tape. ("Outputs \( f(w) \)"")
Computable Functions

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Example 1: \( f((x, y)) = x + y \)

Example 2: \( f((M, w)) = \langle M' \rangle \) where \( M \) is a TM, \( w \) is a string, and \( M' \) is a TM that ignores its input and simulates running \( M \) on \( w \).
Mapping Reductions

Definition: \( A, B \subseteq \Sigma^* \)

Language \( A \) is mapping reducible to language \( B \), written \( A \leq_m B \)

if there is a computable function \( f: \Sigma^* \to \Sigma^* \) such that for all strings \( w \in \Sigma^* \), we have \( w \in A \iff f(w) \in B \)
Mapping Reductions

Definition:
Language $A$ is mapping reducible to language $B$, written $A \leq_m B$ if there is a computable function $f : \Sigma^* \rightarrow \Sigma^*$ such that for all strings $w \in \Sigma^*$, we have $w \in A \iff f(w) \in B$.

If $A \leq_m B$, which of the following is true?

a) $\overline{A} \leq_m B$

b) $A \leq_m \overline{B}$

c) $\overline{A} \leq_m \overline{B}$

d) $\overline{B} \leq_m \overline{A}$
Decidability

Theorem: If $A \leq_m B$ and $B$ is decidable, then $A$ is also decidable

Proof: Let $M$ be a decider for $B$ and let $f: \Sigma^* \rightarrow \Sigma^*$ be a mapping reduction from $A$ to $B$. Construct a decider for $A$ as follows:

Proof of correctness ($N$ decides $A$):

1. If $w \in A$, then $f(w) \in B$ [def of mapping red.]
   $\Rightarrow M$ accepts $f(w)$ [M decides $B$]
   $\Rightarrow N$ accepts

2. If $w \notin A$, then $f(w) \notin B$ [def of mapping red.]
   $\Rightarrow M$ rejects $f(w)$ [M decides $B$]
   $\Rightarrow N$ rejects

On input $w$:
1. Compute $f(w)$
2. Run $M$ on input $f(w)$
3. If $M$ accepts, accept. If it rejects, reject.
Undecidability

Theorem: If $A \leq_m B$ and $B$ is decidable, then $A$ is also decidable

(Contrapositive of Thm)

Corollary: If $A \leq_m B$ and $A$ is undecidable, then $B$ is also undecidable
Old Proof: Equality Testing for TMs

\[ EQ_{TM} = \{ \langle M_1, M_2 \rangle \mid M_1, M_2 \text{ are TMs and } L(M_1) = L(M_2) \} \]

**Theorem:** \( EQ_{TM} \) is undecidable

**Proof:** Suppose for contradiction that there exists a decider \( R \) for \( EQ_{TM} \). We construct a decider for \( E_{TM} \) as follows:

On input \( \langle M \rangle \):

1. Construct TMs \( M_1, M_2 \) as follows:
   
   \[ M_1 = M \]
   
   \[ M_2 = "\text{On input } x, 1. \text{Ignore } x \text{ and reject}" \]

2. Run \( R \) on input \( \langle M_1, M_2 \rangle \)

3. If \( R \) accepts, accept. Otherwise, reject.

This is a reduction from \( E_{TM} \) to \( EQ_{TM} \)
New Proof: Equality Testing for TMs

\[ \text{EQ}_\text{TM} = \{ \langle M_1, M_2 \rangle \mid M_1, M_2 \text{ are TMs and } L(M_1) = L(M_2) \} \]

Theorem: \( E_{\text{TM}} \leq_m \text{EQ}_\text{TM} \) hence \( \text{EQ}_\text{TM} \) is undecidable

Proof: The following TM \( N \) computes the reduction \( f: \)

\[ f : \Sigma^* \rightarrow \Sigma^* \quad \text{Input: } \langle m \rangle \quad \langle m \rangle \in E_{\text{TM}} \Leftrightarrow \langle m, m \rangle \in \text{EQ}_\text{TM} \]

On input \( \langle M \rangle \):

1. Construct TMs \( M_1, M_2 \) as follows:
   
   \[
   M_1 = M \\
   M_2 = \text{"On input } x, \text{ 1. Ignore } x \text{ and reject"}
   \]

2. Output \( \langle M_1, M_2 \rangle \)
Mapping Reductions: Recognizability

Theorem: If $A \leq_m B$ and $B$ is recognizable, then $A$ is also recognizable.

Proof: Let $M$ be a recognizer for $B$ and let $f : \Sigma^* \rightarrow \Sigma^*$ be a mapping reduction from $A$ to $B$. Construct a recognizer for $A$ as follows:

1. Compute $f(w)$
2. Run $M$ on input $f(w)$
3. If $M$ accepts, accept. Otherwise, reject.

Correctness:

1. If $w \in A \implies f(w) \in B$ [if $f$ is a mapping red.]
   $\implies M$ accepts [if $M$ recognizes $B$]
   $\implies N$ accepts $w$.

2. If $w \notin A \implies f(w) \notin B$ [if $f$ is a mapping red.]
   $\implies M$ either rejects or loops [if $M$ recognizes $B$]
   $\implies N$ either rejects or loops on $w$. 
Unrecognizability

Theorem: If $A \leq_m B$ and $B$ is recognizable, then $A$ is also recognizable.

Corollary: If $A \leq_m B$ and $A$ is unrecognizable, then $B$ is also unrecognizable.

We know $\overline{A_{TM}}$ is unrecognizable.

Corollary: If $\overline{A_{TM}} \leq_m B$, then $B$ is unrecognizable.

Corollary: If $A_{TM} \leq_m \overline{B}$ then $B$ is unrecognizable.
Recognizability and $A_{TM}$

Let $L$ be a language. Which of the following is true?

- **a)** If $L \leq_m A_{TM}$, then $L$ is recognizable
- **b)** If $A_{TM} \leq_m L$, then $L$ is recognizable
- **c)** If $L$ is recognizable, then $L \leq_m A_{TM}$
- **d)** If $L$ is recognizable, then $A_{TM} \leq_m L$

**Theorem:** $L$ is recognizable if and only if $L \leq_m A_{TM}$
Recognizability and $A_{TM}$

**Theorem:** $L$ is recognizable if and only if $L \leq_m A_{TM}$

**Proof:**

$\leq_m$ If $L \leq_m A_{TM}$, then by Thm 9.4 that $A_{TM}$ is recognizable, $L$ is recognizable.

$\Rightarrow$ Suppose $L$ is recognizable by TM $M$.

Claim:Exists a mapping reduction $f$ from $L$ to $A_{TM}$.

Want $w \in L \iff f(w) \in A_{TM}$. Compute $f$ using the following TM $R$:

"On input $w$:

Output $\langle M, w \rangle$.

Correctness: $w \in L \Rightarrow M$ accepts $w \Rightarrow \langle M, w \rangle \in A_{TM}$

$w \notin L \Rightarrow M$ does not accept $w \Rightarrow \langle M, w \rangle \notin A_{TM}$
Example: Another reduction to $EQ_{TM}$

$EQ_{TM} = \{ \langle M_1, M_2 \rangle | M_1, M_2 \text{ are TMs and } L(M_1) = L(M_2) \}$

Theorem: $A_{TM} \leq_m EQ_{TM}$  \[ A_{TM} = \{ \langle m, w \rangle \mid TM \text{ M accepts } w \} \]

Proof: The following TM $N$ computes the reduction $f$:

$\langle m, w \rangle \in A_{TM} \Rightarrow f(\langle m, w \rangle) = \langle m, m_2 \rangle \in EQ_{TM}$

$\langle m, w \rangle \notin A_{TM} \Rightarrow f(\langle m, w \rangle) = \langle m, m_2 \rangle \notin EQ_{TM}$

What should the inputs and outputs to $f$ be?

a) $f$ should take as input a pair $\langle M_1, M_2 \rangle$ and output a pair $\langle M, w \rangle$

b) $f$ should take as input a pair $\langle M, w \rangle$ and output a pair $\langle M_1, M_2 \rangle$

c) $f$ should take as input a pair $\langle M_1, M_2 \rangle$ and either accept or reject

d) $f$ should take as input a pair $\langle M, w \rangle$ and either accept or reject
Example: Another reduction to $EQ_{TM}$

$EQ_{TM} = \{ \langle M_1, M_2 \rangle | M_1, M_2 \text{ are TMs and } L(M_1) = L(M_2) \}$

Theorem: $A_{TM} \leq_m EQ_{TM}$

Proof: The following TM computes the reduction:

On input $\langle M, w \rangle$:
1. Construct TMs $M_1, M_2$ as follows:
   \[
   M_1 = \text{"On input } x, \quad M_2 = \text{"On input } x, \\
   1. \text{Ignore } x \\
   2. \text{Run } M \text{ on } w, \text{ if accept, accept. If reject, reject.}" \\
   \]
2. Output $\langle M_1, M_2 \rangle$

\[
\begin{align*}
L(M_1) &\colon \begin{cases} \sum^\ast & \text{if } M \text{ accepts } w \\
\emptyset & \text{if } M \text{ does not accept } w \end{cases} \\
L(M_2) &\colon \sum^\ast
\end{align*}
\]

Correctness of reduction:
1. If $\langle M, w \rangle \in A_{TM} \Rightarrow L(M_1) = \sum^\ast \\
2. If $\langle M, w \rangle \notin A_{TM} \Rightarrow L(M_1) = \emptyset \neq \sum^\ast = L(M_2)$
Consequences of $A_{TM} \leq_m EQ_{TM}$

1. Since $A_{TM}$ is undecidable, $EQ_{TM}$ is also undecidable

2. $A_{TM} \leq_m EQ_{TM}$ implies $\overline{A_{TM}} \leq_m \overline{EQ_{TM}}$

Since $\overline{A_{TM}}$ is unrecognizable, $\overline{EQ_{TM}}$ is unrecognizable
$EQ_{TM}$ itself is also unrecognizable

$EQ_{TM} = \{ \langle M_1, M_2 \rangle | M_1, M_2 \text{ are TMs and } L(M_1) = L(M_2) \}$

Theorem: $\overline{A_{TM}} \leq_m EQ_{TM}$ hence $EQ_{TM}$ is unrecognizable

Proof: The following TM computes the reduction:

On input $\langle M, w \rangle$:

1. Construct TMs $M_1, M_2$ as follows:
   
   $M_1 = \text{"On input } x, \text{ 1. Ignore } x \text{ 2. Run } M \text{ on input } w \text{ 3. If } M \text{ accepts, accept. Otherwise, reject."}$
   
   $M_2 = \text{"On input } x, \text{ 1. Ignore } x \text{ and reject"}$

2. Output $\langle M_1, M_2 \rangle$