BU CS 332 – Theory of Computation

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Lecture 22:

- NP

Reading:

Sipser Ch 7.3-7.4

HW 9

Wednesday 11:59 PM

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Nondeterministic time and NP

Let $f : \mathbb{N} \to \mathbb{N}$

A NTM M runs in time $f(n)$ if on every input $w \in \Sigma^n$,
M halts on w within at most $f(n)$ steps on every computational branch

$\text{NTIME}(f(n))$ is a class (i.e., set) of languages:

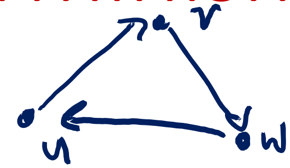
A language $A \in \text{NTIME}(f(n))$ if there exists an NTM M that

1) Decides A, and
2) Runs in time $O(f(n))$

Definition: NP is the class of languages decidable in polynomial time on a nondeterministic TM

$$\text{NP} = \bigcup_{k=1}^{\infty} \text{NTIME}(n^k)$$

Speeding things up with nondeterminism



HW 5 Problem 3:

$$TRIANGLE = \{\langle G \rangle \mid \text{digraph } G \text{ contains a triangle}\}$$

Deterministic algorithm: $G = (V, E)$

For each $u \in V$:
  For each $v \in V$:
    For each $w \in V$:
      Check: $(u,v), (v,w), (W,u) \in E$

Runtime $\geq |V|^3$

vertices $= |V|$
$V = \{1, \ldots, k\}$
Writing down a # between 1 and k takes $\log_2 k$ bits

Nondeterministic algorithm:

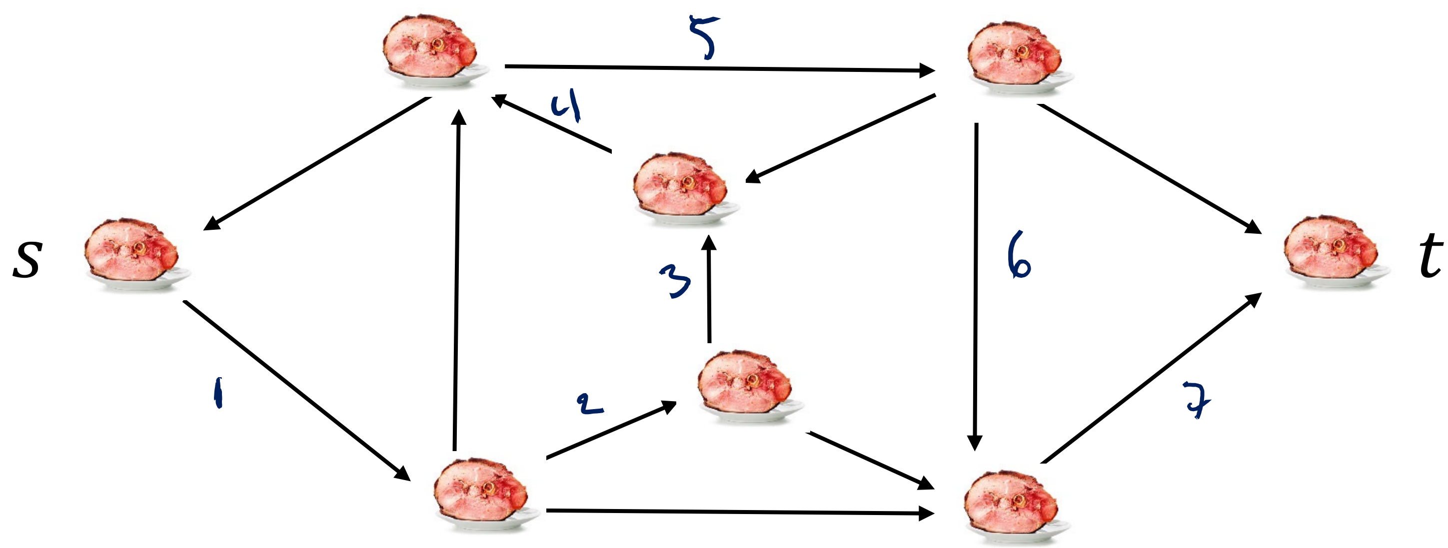
1) Nondeterministically guess (u, v, w)

Guess $O(\log|V|)$ bits

2) Check if $(u,v), (v,w), (w,u) \in E$

Hamiltonian Path

$HAMPATH = \{\langle G, s, t\rangle \mid G$ is a directed graph and there is a path from s to t that passes through every vertex exactly once$\}$



$HAMPATH \in \mathrm{NP}$

The following **nondeterministic** algorithm decides $HAMPATH$ in polynomial time:

On input $\langle G, s, t\rangle$: (Vertices of G are numbers $1, \dots, k$)

1. **Nondeterministically** guess a sequence $c_1, c_2, \dots, c_k$ of numbers $1, \dots, k$
2. Check that $c_1, c_2, \dots, c_k$ is a permutation: Every number $1, \dots, k$ appears exactly once
3. Check that $c_1 = s$, $c_k = t$, and there is an edge from every c_i to c_{i+1}
4. Accept if all checks pass, otherwise, reject.

Analyzing the algorithm

Need to check:

1) Correctness

a) If $\langle G,s,t\rangle \in HAMPATH$, $\exists$ a Hamiltonian path $c_1,\dots,c_k$
$\Rightarrow$ Branch of computation that guesses this path leads NTM to accept ✓

b) If $\langle G,s,t\rangle \notin HAMPATH$, then every $c_1,\dots,c_k$ is not a Hamiltonian path from s to t. $\Rightarrow$ Every computation branch rejects ✓

2) Running time

Guessing $c_1,\dots,c_k$ takes $O(k \log k)$ time

Checking permutation takes $O((k\log k)^2)$ time

Checking $c_1,\dots,c_k$ is an s.t path takes $O(k)$ lookups to adj. matrix / adj. list of G

An alternative characterization of $\mathbf{NP}$

"Languages with polynomial-time verifiers"

How did we design an NTM for HAMPATH?

- Given a candidate path, it is easy (poly-time) to check whether this path is a Hamiltonian path
- We designed a poly-time NTM by nondeterministically guessing this path and then checking it
- Lots of problems have this structure (CLIQUE, 3-COLOR, COMPOSITE,…). They might be hard to solve, but a candidate solution is easy to check.

An alternative characterization of NP

"Languages with polynomial-time verifiers"

A verifier for a language L is a deterministic algorithm V such that $w \in L$ iff there exists a string c such that $V(\langle w, c \rangle)$ accepts

"instance"

"certificate" "witness" "proof"

Running time of a verifier is only measured in terms of $|w|$

V is a polynomial-time verifier if it runs in time polynomial in $|w|$ on every input $\langle w, c \rangle$

(Without loss of generality, $|c|$ is polynomial in $|w|$, i.e., $|c| = O(|w|^k)$ for some constant k)

$HAMPATH$ has a polynomial-time verifier

Certificate c: $c\,(c_1, \ldots, c_n)$ a candidate path

Runtime: A valid path c has length poly. in $|w|$

Verifier V:

From before: checking c takes poly time

On input $\langle \underbrace{G, s, t}_{w}; \underbrace{c}_{\text{certificate}} \rangle$: (Vertices of G are numbers $1, \ldots, k$)

1. Check that $c_1, c_2, \ldots, c_k$ is a permutation: Every number $1, \ldots, k$ appears exactly once
2. Check that $c_1 = s$, $c_k = t$, and there is an edge from every c_i to c_{i+1}
3. Accept if all checks pass, otherwise, reject.

Correctness:

a) If $\langle G, s, t \rangle \in L$, $\exists$ Ham path $c_1, \ldots, c_n$ from s to t $\Rightarrow \exists\, c$ s.t. $V(\langle G, s, t; c \rangle)$ accepts

b) If $\langle G, s, t \rangle \notin L$, then every c is not a Hampath $\Rightarrow \forall c$ $V(\langle G, s, t; c \rangle)$ rejects

NP is the class of languages with polynomial-time verifiers

Theorem: A language $L \in \mathrm{NP}$ iff there is a polynomial-time verifier for L

Alternative proof of $\mathrm{NP} \subseteq \mathrm{EXP}$

input $\langle w, c \rangle$

One can prove $\mathrm{NP} \subseteq \mathrm{EXP}$ as follows. Let V be a verifier for a language L running in time $T(n)$. We can construct a $2^{O(T(n))}$ time algorithm for L as follows.

input w

~~a)~~ On input $\langle w, c \rangle$, run V on $\langle w, c \rangle$ and output the result

b) On input w, run V on all possible $\langle w, c \rangle$, where c is a certificate. Accept if any run accepts.

$\boxed{\text{c)}}$ On input w, run V on all possible $\langle w, c \rangle$, where c is a certificate of length at most $T(|w|)$. Accept if any run accepts.

d) On input w, run V on all possible $\langle x, c \rangle$, where x is a string of length $|w|$ and c is a certificate of length at most $T(|w|)$. Accept if any run accepts.

NP is the class of languages with polynomial-time verifiers

Theorem: A language $L \in \mathrm{NP}$ iff there is a polynomial-time verifier for L

Proof: $\Leftarrow$ Let L have a $\overbrace{\text{poly-time}}^{\text{time } T(n)}$ verifier $V(\langle w, c\rangle)$

Idea: Design NTM N for L that nondeterministically guesses a certificate

NTM N:

On input w:

1) Nondet. guess c of length $\le T(|w|)$
2) Run $V(\langle w, c\rangle)$. If accepts, accept. If rejects, reject

Correctness:

$w \in L \iff \exists\, c,\ |c| \le T(|w|)$ s.t. $V(\langle w, c\rangle)$ accepts

$\iff N$ accepts w on some comp. branch

Run time:

$|c|$ is polynomial in $|w|$ $\Rightarrow$ step 1 is poly time

V runs in poly-time $\Rightarrow$ step 2 poly-time

NP is the class of languages with polynomial-time verifiers

$\Rightarrow$ Let L be decided by an NTM N running in time $T(n)$ and making up to b nondeterministic choices in each step

Idea: Design verifier V for L where certificate is sequence of "good" nondeterministic choices

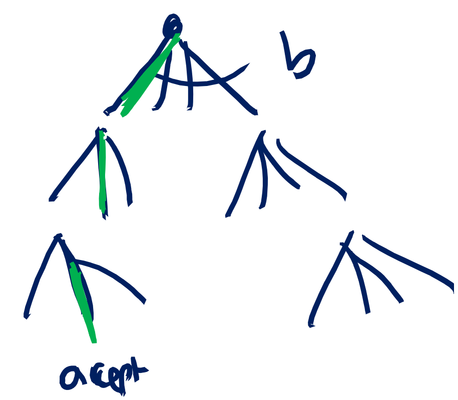
Certificate: $c = (c_1, \ldots, c_{T(n)}) \in [b]^{T(n)}$

c_i is the "good" nondet. choice @ time step i

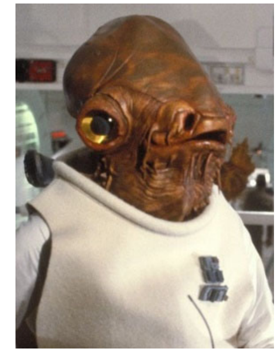
$V(\langle w, c \rangle)$:

1) Simulate N on input w, where at every time step i, make nondet. choice c_i

2) If simulation reaches accept, accept
reject, reject.



WARNING: Don't mix-and-match the NTM and verifier interpretations of $\mathbf{NP}$



To show a language L is in NP, do exactly one:

1) Exhibit a poly-time NTM for L

   N = "On input w:

   <Do some nondeterministic stuff>…"

OR

2) Exhibit a poly-time (deterministic) verifier for L

   V = "On input w and certificate c:

   <Do some deterministic stuff>…"

Examples of NP languages: SAT

"Is there an assignment to the variables in a logical formula that make it evaluate to true?"

- Boolean variable: Variable that can take on the value true/false (encoded as 0/1) Ex: x_1, x_2, x_3 x, y, z
- Boolean operations: $\wedge$ (AND), $\vee$ (OR), $\neg$ (NOT)
- Boolean formula: Expression made of Boolean variables and operations. Ex: $(x_1 \vee \overline{x_2}) \wedge x_3$ Ex: $x_1 = 1$, $x_2 = 1$, $x_3 = 1$
  $\quad 1 \quad 0 \quad 1$
- An assignment of 0s and 1s to the variables satisfies a formula φ if it makes the formula evaluate to 1 (1,1,1) is a satisfying ass'mt
- A formula φ is satisfiable if there exists an assignment that satisfies it to $(x_1 \vee \overline{x_2}) \wedge x_3$

Examples of **NP** languages: SAT

Ex: $(x_1 \vee \overline{x_2}) \wedge x_3$ Yes because $(1,1,1)$ is a sat assm't   Satisfiable?

Ex: $(x_1 \vee x_2) \wedge \overline{x_1} \wedge \overline{x_2}$ $= (x_1 \vee x_2) \wedge \overline{(x_1 \vee x_2)}$ Satisfiable?

Not satisfiable (no possible sat. assm't)

$$SAT = \{\langle \varphi \rangle \mid \varphi \text{ is a satisfiable formula}\}$$

Claim: $SAT \in \mathrm{NP}$

NTM for SAT

On input $\varphi(x_1, \ldots, x_m)$:

1) Nondet. guess $(c_1, \ldots, c_m) \in \{0,1\}^m$
2) Evaluate $\varphi(c_1, \ldots, c_m)$.
   If True: accept
   False: reject

Poly-time verifier for SAT:

Certificate $c = (c_1, \ldots, c_m) \in \{0,1\}^m$

On input $\langle \varphi, c \rangle$:

1) Evaluate $\varphi(c_1, \ldots, c_m)$.
   If True: accept
   If False: reject