CS 332: *Theory of Computation* Prof. Mark Bun
Boston University September 22, 2022

Homework 3 – Due Thursday, September 29, 2022 at 11:59 PM

Reminder Collaboration is permitted, but you must write the solutions *by yourself without assistance*, and be ready to explain them orally to the course staff if asked. You must also identify your collaborators and write "Collaborators: none" if you worked by yourself. Getting solutions from outside sources such as the Web or students not enrolled in the class is strictly forbidden.

Problems There are 6 required problems and one bonus problems.

1. (**Closure properties**)
   (a) Given languages A, B, define the language $MIX(A, B)$ by
   $$MIX(A, B) = \{x_1y_1x_2y_2 \dots x_ny_n \mid n \geq 0, x_i \in A, y_i \in B\}.$$
   Note that each x_i, y_i is a *string*. Show that the class of regular languages is closed under MIX. Hint: You don't need to construct an NFA recognizing $MIX(A, B)$ if you can find a way to express it in terms of other operations.

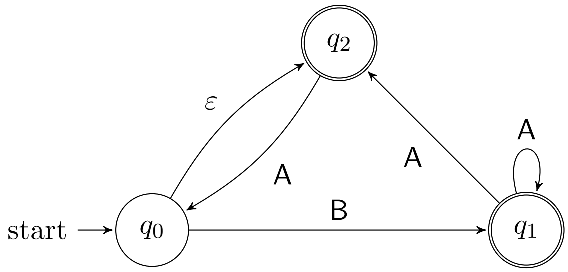
   (b) Given a language A over alphabet Σ, define the language $TAIL(A) = \{y \in \Sigma^* \mid xy \in A \text{ for some } x \in \Sigma^*\}$. Show that the regular languages are closed under $TAIL$.

2. (**Regex to description**) Give plain English descriptions of the languages generated by each of the following regular expressions
   (a) $(a \cup b)^* \cup c^*$
   (b) $1(000)^*1$
   (c) $a(ba)^*b$
   (d) $\emptyset^*$
   (e) $(\emptyset \cup \varepsilon)^*$

3. (**Regular expressions vs. finite automata**) Please log on to AutomataTutor to submit solutions for this question.
   (a) (**Description to regex**) Give regular expressions generating the following languages:
      i. $\{w \in \{0,1\}^* \mid w \text{ has exactly two 0's and at least one 1}\}$
      ii. $\{w \in \{0,1\}^* \mid w \text{ is not the string } 01\}$
      iii. $\{w \in \{0,1\}^* \mid \text{the number of 1's in } w \text{ is divisible by } 3\}$.

   (b) (**Regex to NFA**) Use the procedure described in class (also in Sipser, Lemma 1.55) to convert $(\mathsf{AT} \cup \mathsf{TA} \cup \mathsf{CG} \cup \mathsf{GC})^*$ to an equivalent NFA. Simplify your NFA.

   (c) (**NFA to regex**) Convert the following NFA to an equivalent regular expression.



4. (**Conversion procedures as algorithms**) Consider the following pseudocode describing an algorithm taking as input a regex and outputting the description of an equivalent NFA.

RegexToNFA(R)
Input : Regular expression R
Output: Equivalent NFA N
if $R = \emptyset$ **then**
  return NFA.emptyLanguage();
else if $R = \varepsilon$ **then**
  return NFA.emptyString();
else if $R = a$ **then**
  return NFA.symbol(a);
else if $R = R_1 \cup R_2$ **then**
  return NFA.union(RegexToNFA(R_1), RegexToNFA(R_2));
else if $R = R_1 \circ R_2$ **then**
  return NFA.concatenate(RegexToNFA(R_1), RegexToNFA(R_2));
else if $R = R_1^*$ **then**
  return NFA.star(RegexToNFA(R_1));

Here, you can assume that the subroutines NFA.emptyLanguage(), NFA.emptyString(), and NFA.symbol(a) return NFAs recognizing the languages $\emptyset, \{\varepsilon\}, \{a\}$, respectively, as described in Sipser's proof of Lemma 1.55 or in Lecture 6, slide 5. Moreover, NFA.union(N_1, N_2) takes as input two NFAs and outputs the NFA recognizing $L(N_1) \cup L(N_2)$ described in Sipser's proof of Theorem 1.45, and similarly for NFA.concatenate and NFA.star.

(a) If N_1 and N_2 are NFAs with s_1 and s_2 states, respectively, how many states does NFA.union(N_1, N_2) have? How about NFA.concatenate(N_1, N_2)? NFA.star(N_1)?

(b) Define the *size* of a regular expression R to be the number of appearances of $\emptyset, \varepsilon, \cup, \circ, ^*$ and alphabet symbols in R. If R is a regular expression of size 1, what is the maximum number of states in RegexToNFA(R)?

(c) For a natural number k, let $S(k)$ be the maximum number of states RegexToNFA(R) can have over all regexes R of size k. Prove by (strong) induction on k that $S(k) \leq 2k$.

Now consider the following pseudocode describing an algorithm taking as input an NFA and outputting an equivalent regex.

(d) Suppose the starting NFA N has exactly one symbol labeling each transition present in its state diagram. (This simplifying assumption makes the calculations cleaner, and in particular, independent of the alphabet size.)

NFAtoRegex(N)
Input : NFA N
Output: Equivalent regular expression R
$M_0 \leftarrow$ NFAtoGNFA(N);
$k \leftarrow$ number of states of M_0;
for $i \leftarrow 1$ **to** $k-2$ **do**
| Obtain M_i from M_{i-1} by ripping out state q_i and updating transitions appropriately;
end
return the regex labeling the transition from q_0 to q_{accept} in M_{k-2};

Let $\ell(i)$ be the maximum possible size of a regular expression appearing on any transition in M_i. Prove by induction on i that $\ell(i) \leq 4^{i+1} - 3$.

(e) Show that if N is an NFA with s states, then NFAtoRegex(N) is a regular expression of size at most 4^{s+1}.

5. (**Distinguishing set method**)

(a) Let $REP_2 = \{ww \mid w \in \{0,1\}^2\}$. Show that $S = \{00, 01, 10, 11\}$ is pairwise distinguishable by REP_2. That is, for every pair $x, y \in S$, argue that there is a string z such that exactly one of xz and yz is in REP_2.

(b) What does part (a) tell you about the smallest number of states a DFA recognizing REP_2 can have? Explain your answer.

(c) For any $k \geq 1$, let $REP_k = \{ww \mid w \in \{0,1\}^k\}$. Show that every DFA recognizing REP_k requires at least 2^k states.

(d) Show that every NFA recognizing REP_k requires at least k states.

6. (**Individual Review: NO COLLABORATION PERMITTED**) Let S be the set of all languages L such that every string in L has length at least 2. Define the operation swap(L) that produces a new language by swapping the first and last characters of every string in L. For example, swap($\{aab, bbb, abbab\}$) = $\{baa, bbb, bbbaa\}$.

For each of the following statements, either provide a proof or give a counterexample and justify why it's a counterexample.

(a) For all $L \in S$, swap(L) $\in S$.

(b) For all $L \in S$, swap(L^R) = (swap(L))^R.

(c) For all $L_1, L_2, \in S$, swap($L_1 \circ L_2$) = swap(L_2) $\circ$ swap(L_1).

7. (**Bonus Problem**) Prove that for every natural number n, there is a language B_n such that a) B_n is recognizable by an NFA with $n+1$ states, but b) If $B_n = A_1 \cup \cdots \cup A_k$ for regular languages $A_1, \ldots, A_k$, then at least one of the languages A_i requires a DFA with at least $2^{n/k}$ states.