BU CS 332 – Theory of Computation

Link to polls:
https://forms.gle/exHaBu5mQgR57NDq5

Lecture 2:

- Parts of a Theory of Computation
- Sets, Strings, and Languages
- Start Finite Automata?

Reading:

Sipser Ch 0

Reminders:
- HW0 due tonight, HW1 out
- Lecture 1 check-in due on Gradescope

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What makes a good theory?

- General ideas that apply to many different systems
- Expressed simply, abstractly, and precisely

Parts of a Theory of Computation

- Models for machines (computational devices)
- Models for the problems machines can be used to solve
- Theorems about what kinds of machines can solve what kinds of problems, and at what cost

What is a (Computational) Problem?

For us: A problem will be the task of recognizing whether a *string* is in a *language*

E.g. Parity: Given a string of a's and b's, does it contain an even number of a's?

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- **Alphabet**: A finite set Σ Ex. $\Sigma = \{\mathrm{a}, \mathrm{b}\}$
- **String:** A finite concatenation of alphabet symbols

  Ex. $\mathrm{bba}, \mathrm{ababb}$

  ε denotes empty string, length 0

  Σ^* = set of all strings using symbols from Σ

  Ex. $\{\mathrm{a}, \mathrm{b}\}^* = \{\varepsilon, \mathrm{a}, \mathrm{b}, \mathrm{aa}, \mathrm{ab}, \mathrm{ba}, \mathrm{bb}, \dots\}$
- **Language:** A set $L \subseteq \Sigma^*$ of strings

Examples of Languages

Parity: Given a string consisting of a's and b's, does it contain an even number of a's?

Σ = L =

Primality: Given a natural number x (represented in binary), is x prime?

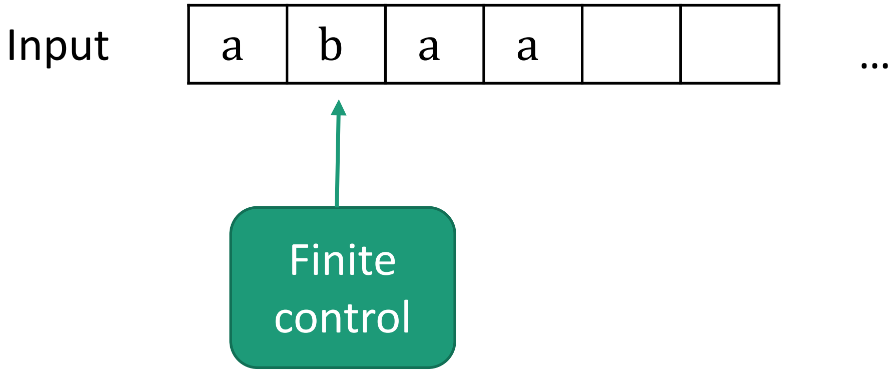
Σ = L =

Halting Problem: Given a C program, can it ever get stuck in an infinite loop?

Σ = L =

Machine Models

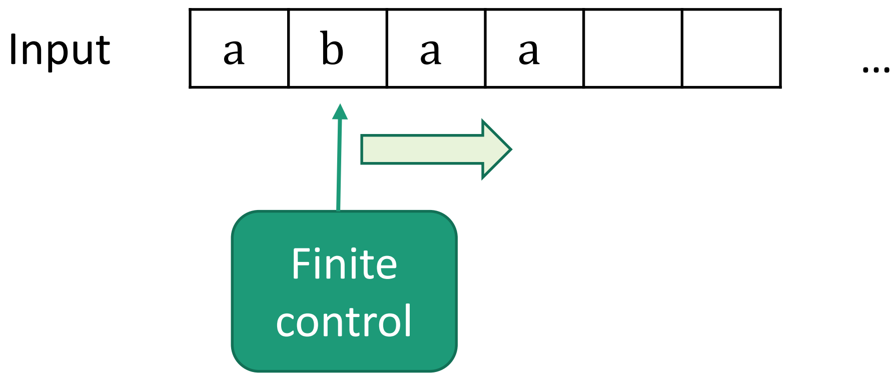
Computation is the processing of information by the **unlimited application** of a **finite set** of operations or rules



Abstraction: We don't care how the control is implemented. We just require it to have a finite number of states, and to transition between states using fixed rules.

Machine Models

- Finite Automata (FAs): Machine with a finite amount of unstructured memory

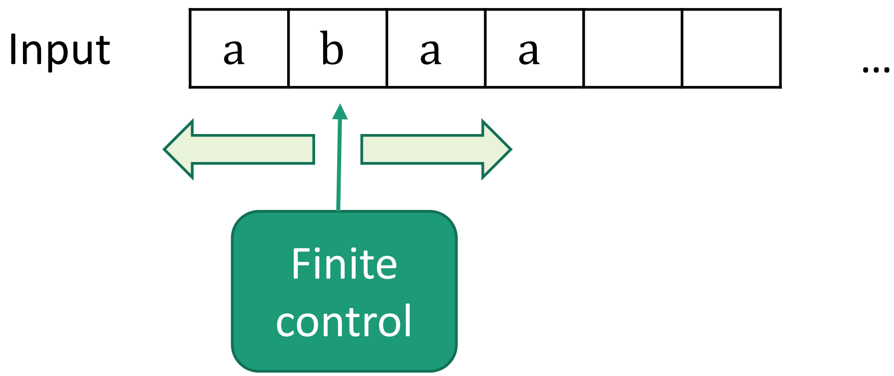


Control scans left-to-right
Can check simple patterns
Can't perform unlimited counting

Useful for modeling chips, simple control systems, choose-your-own adventure games…

Machine Models

- Turing Machines (TMs): Machine with unbounded, unstructured memory



Control can scan in both directions
Control can both read and write

Model for general sequential computation

Church-Turing Thesis: Everything we intuitively think of as "computable" is computable by a Turing Machine

What theorems would we like to prove?

We will define classes of languages based on which machines can recognize them

Inclusion: Every language recognizable by a FA is also recognizable by a TM

Non-inclusion: There exist languages recognizable by TMs which are not recognizable by FAs

Completeness: Identify a “hardest” language in a class

Robustness: Alternative definitions of the same class

Ex. Languages recognizable by FAs = regular expressions

Why study theory of computation?

- You'll learn how to formally reason about computation
- You'll learn the technology-independent foundations of CS

Philosophically interesting questions:

- Are there well-defined problems which cannot be solved by computers?
- Can we always find the solution to a puzzle faster than trying all possibilities?
- Can we say what it means for one problem to be "harder" than another?

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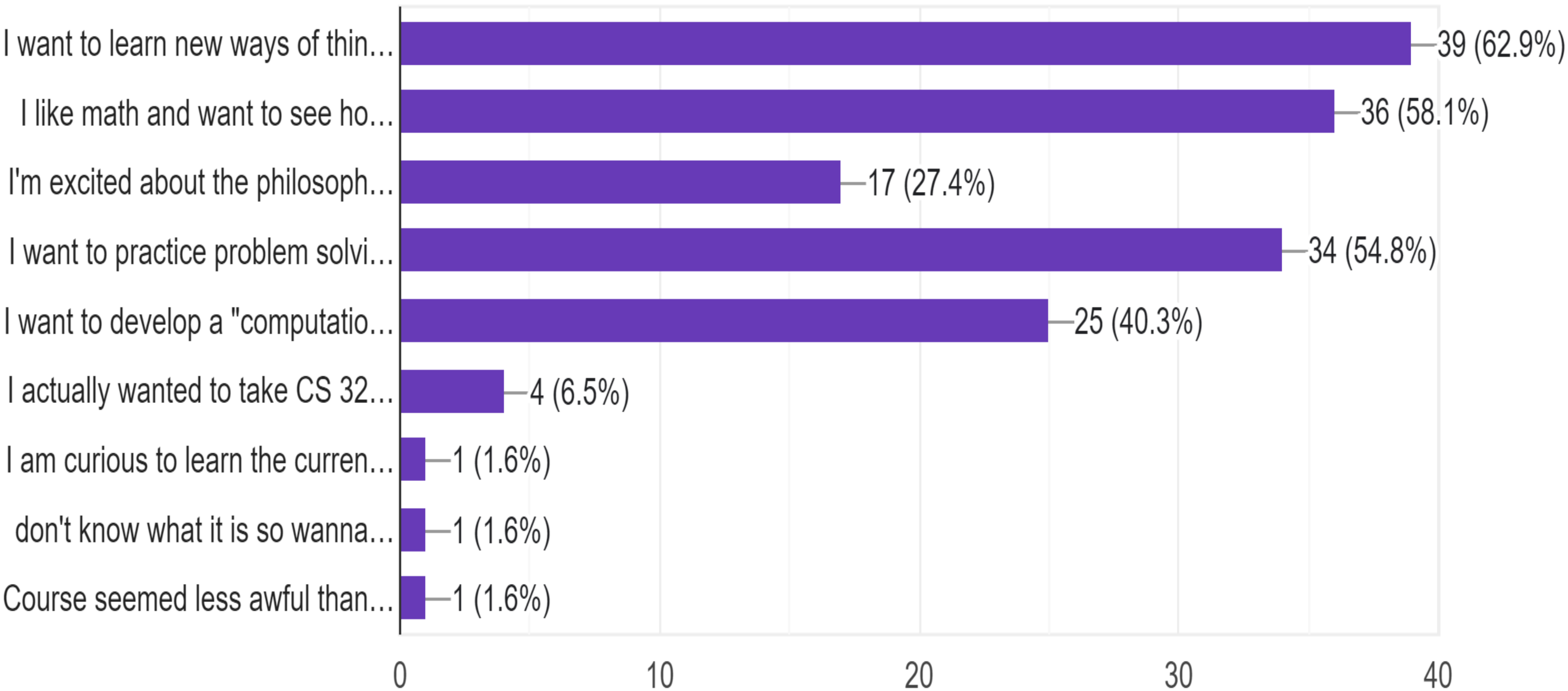
Connections to other parts of science:

- Finite automata arise in compilers, AI, coding, chemistry
https://cstheory.stackexchange.com/a/14818
- Hard problems are essential to cryptography
- Computation occurs in cells/DNA, the brain, economic systems, physical systems, social networks, etc.

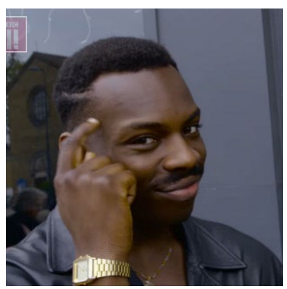
What appeals to you about the theory of computation?



Why study theory of computation?

Practical knowledge for developers


"Boss, I can't find an efficient algorithm.
I guess I'm just too dumb."

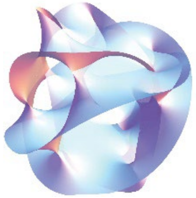
"Boss, I can't find an efficient algorithm
because no such algorithm exists."

Will you be asked about this material on job interviews?

No promises, but a true story…

More about strings and languages

String Theory



- **Symbol:** Ex. $\mathrm{a, b, 0, 1}$
- **Alphabet**: A finite set Σ Ex. $\Sigma = \{\mathrm{a, b}\}$
- **String:** A finite concatenation of alphabet symbols

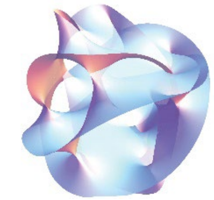
  Ex. $\mathrm{bba, ababb}$

  ε denotes empty string, length 0

  Σ^* = set of all strings using symbols from Σ

  Ex. $\{\mathrm{a, b}\}^* = \{\varepsilon, \mathrm{a, b, aa, ab, ba, bb}, \ldots\}$
- **Language:** A set $L \subseteq \Sigma^*$ of strings

String Theory



- **Length** of a string, written $|x|$, is the number of symbols

Ex. $|\mathrm{abba}| =$ $|\varepsilon| =$

- **Concatenation** of strings x and y, written xy, is the symbols from x followed by the symbols from y

Ex. $x = \mathrm{ab}, y = \mathrm{ba}$ $\Rightarrow$ $xy =$

$x = \mathrm{ab}, y = \varepsilon$ $\Rightarrow$ $xy =$

- **Reversal** of string x, written x^R, consists of the symbols of x written backwards

Ex. $x = \mathrm{aab}$ $\Rightarrow$ $x^R =$

Fun with String Operations

What is $(xy)^R$?

Ex. $x = \text{aba}, y = \text{bba} \quad \Rightarrow \quad xy =$

$\Rightarrow \quad (xy)^R =$

a) $x^R y^R$

b) $y^R x^R$

c) $(yx)^R$

d) xy^R

Fun Proofs with String Operations

Claim: $(xy)^R =$

Proof: Let $x = x_1 x_2 \ldots x_n$ and $y = y_1 y_2 \ldots y_m$

Then $(xy)^R =$

Not even the most formal way to do this:

1. Define string length recursively
2. Prove by induction on $|y|$

Languages

A language L is a set of strings over an alphabet Σ

i.e., $L \subseteq \Sigma^*$

Languages = computational (decision) problems

Input: String $x \in \Sigma^*$

Output: Is $x \in L$? (YES or NO?)

Some Simple Languages

$\Sigma = \{0, 1\}$ $\Sigma = \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$

$\emptyset$ (Empty set)

Σ^* (All strings)

$\Sigma^n = \{x \in \Sigma^* \mid |x| = n\}$
(All strings of length n)

Some More Interesting Languages

- $L_1 =$ The set of strings $x \in \{\mathrm{a}, \mathrm{b}\}^*$ that have an equal number of a's and b's

- $L_2 =$ The set of strings $x \in \{\mathrm{a}, \mathrm{b}\}^*$ that start with (0 or more) a's and are followed by an equal number of b's

- $L_3 =$ The set of strings $x \in \{0, 1\}^*$ that contain the substring '0100'

Some More Interesting Languages

- $L_4 =$ The set of strings $x \in \{\mathrm{a}, \mathrm{b}\}^*$ of length at most 4

- $L_5 =$ The set of strings $x \in \{\mathrm{a}, \mathrm{b}\}^*$ that contain at least two a's

New Languages from Old

L_6 = The set of strings $x \in \{\mathrm{a}, \mathrm{b}\}^*$ that have an equal number of a's and b's and length greater than 4

Since languages are just sets of strings, can build them using set operations:

$A \cup B$ "union"

$A \cap B$ "intersection"

$\bar{A}$ "complement"

New Languages from Old

$L_6 =$ The set of strings $x \in \{\mathrm{a}, \mathrm{b}\}^*$ that have an equal number of a's and b's and have length greater than 4

- $L_1 =$ The set of strings $x \in \{\mathrm{a}, \mathrm{b}\}^*$ that have an equal number of a's and b's
- $L_4 =$ The set of strings $x \in \{\mathrm{a}, \mathrm{b}\}^*$ of length at most 4

$$\Rightarrow L_6 =$$

Operations Specific to Languages

- **Reverse**: $L^R = \{x^R \mid x \in L\}$

  Ex. $L = \{\varepsilon, \mathrm{a}, \mathrm{ab}, \mathrm{aab}\}$ $\Rightarrow L^R =$

- **Concatenation:** $L_1 \circ L_2 = \{xy \mid x \in L_1, y \in L_2\}$

  Ex. $L_1 = \{\mathrm{ab}, \mathrm{aab}\}$ $L_2 = \{\varepsilon, \mathrm{b}, \mathrm{bb}\}$

  $\Rightarrow L_1 \circ L_2 =$

A Few “Traps”

String, language, or something else?

ε

$\emptyset$

$\{\varepsilon\}$

$\{\emptyset\}$

Languages

Languages = computational (decision) problems

Input: String $x \in \Sigma^*$

Output: Is $x \in L$? (YES or NO?)

The language **recognized** by a program is the set of strings $x \in \Sigma^*$ that it *accepts*

What Language Does This Program Recognize?

Alphabet $\Sigma = \{\mathrm{a}, \mathrm{b}\}$

On input $x = x_1 x_2 \dots x_n$:

count $= 0$

For $i = 1, \dots, n$:

If $x_i = \mathrm{a}$:

count $=$ count $+ 1$

If count ≤ 4: accept

Else: reject

a) $\{x \in \Sigma^* \mid |x| > 4\}$
b) $\{x \in \Sigma^* \mid |x| \leq 4\}$
c) $\{x \in \Sigma^* \mid |x| = 4\}$
d) $\{x \in \Sigma^* \mid x \text{ has more than } 4 \text{ a}'\text{s}\}$
e) $\{x \in \Sigma^* \mid x \text{ has at most } 4 \text{ a}'\text{s}\}$
f) $\{x \in \Sigma^* \mid x \text{ has exactly } 4 \text{ a}'\text{s}\}$

Deterministic Finite Automata

A (Real-Life?) Example

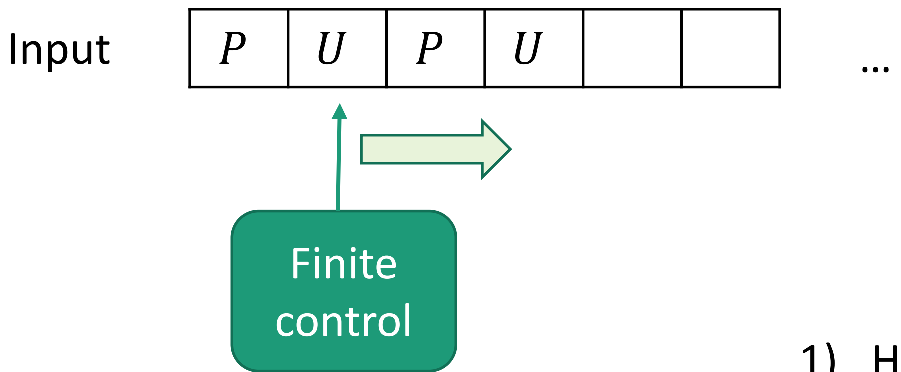
- Example: Kitchen scale
- P = Power button (ON / OFF)
- U = Units button (cycles through g / oz / lb)

  Only works when scale is ON, but units remembered when scale is OFF
- Starts OFF in g mode

- A computational problem: Does a sequence of button presses in $\{P, U\}^*$ leave the scale ON in oz mode?

Machine Models

- <u>Finite Automata (FAs):</u> Machine with a finite amount of unstructured memory



Control scans left-to-right

1) How does the control start?
2) What are the different “states” that the control can be in?
3) When the control reads an new input character, how does it transition to a new state?
4) How do I know if I’m in the desired state at the end?

A DFA for the Kitchen Scale Problem

A DFA Recognizing Parity

The language recognized by a DFA is the set of inputs on which it ends in an "accept" state

Parity: Given a string consisting of a's and b's, does it contain an even number of a's?

$\Sigma = \{a, b\}$ $\quad\quad$ $L = \{w \mid w \text{ contains an even number of } a\text{'s}\}$

Anatomy of a DFA