BU CS 332 – Theory of Computation

https://forms.gle/aCxHoyMogZNUpLw96

Lecture 5:

- Closure Properties
- Regular Expressions

Reading:

Sipser Ch 1.2-1.3

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Last Time

- Nondeterministic Finite Automata
- NFAs vs. DFAs
  - Subset construction: NFA -> DFA

Closure Properties

An Analogy

In algebra, we try to identify operations which are common to many different mathematical structures

Example: The integers $\mathbb{Z} = \{\ldots -2, -1, 0, 1, 2, \ldots\}$ are **closed** under

- Addition: $x + y$ — $3 + (-7) = -4$
- Multiplication: $x \times y$ — $4 \times (-2) = -8$
- Negation: $-x$ — $-(-2) = 2$
- …but NOT Division: $x \,/\, y$ — $2/3$ is not an integer

We'd like to investigate similar closure properties of the class of regular languages

Operations on languages

Let $A, B \subseteq \Sigma^*$ be languages. Define

Σ is finite
Σ^* is infinite

Regular Operations:

- Union: $A \cup B$ $= \{x \mid x \in A \text{ or } x \in B\}$
- Concatenation: $A \circ B = \{xy \mid x \in A, y \in B\}$
- Star: $A^* = \{ w_1 w_2 \ldots w_n \mid n \geq 0 \text{ and } w_i \in A\}$ $= \{\varepsilon\} \cup A \cup A \circ A \cup A \circ A \circ A \cup \ldots$ ($n=0$, $n=1$, $n=2$, $n=3$)

Complement: $\bar{A}$ ($\bar{A} = \{x \in \Sigma^* \mid x \notin A\}$)

Intersection: $A \cap B$

Reverse: $A^R = \{ a_1 a_2 \ldots a_n \mid a_n \ldots a_1 \in A\}$

Theorem: The class of regular languages is closed under all six of these operations, i.e., if A and B are regular, applying any of these operations yields a regular language

Digression: $\{ a^n b^n \mid n \geq 0 \}$ is not regular

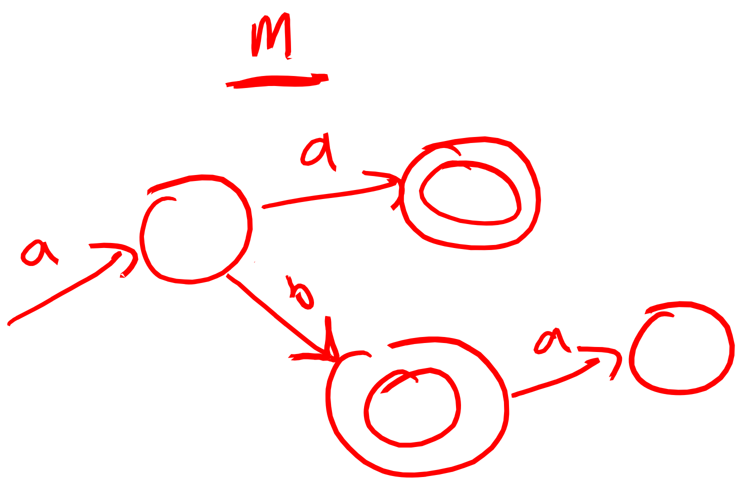
Proving Closure Properties

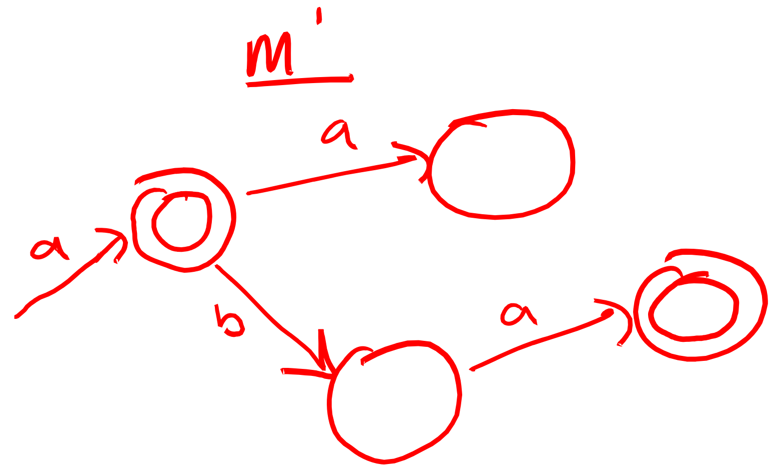
Complement

Complement: $\bar{A} = \{ w \mid w \notin A\}$

Theorem: If A is regular, then $\bar{A}$ is also regular

Proof idea: If A is regular, there exists a DFA M recognizing A. Construct new DFA M' recognizing $\bar{A}$.





Switch accept and non-accept states

Complement, Formally

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA recognizing a language A. Which of the following represents a DFA recognizing $\bar{A}$?

Everything except set of accept states same

a) $(F, \Sigma, \delta, q_0, Q)$

switches role of accept & non-accept

b) $(Q, \Sigma, \delta, q_0, Q \setminus F)$, where $Q \setminus F$ is the set of states in Q that are not in F

"set minus notation"

c) $(Q, \Sigma, \delta', q_0, F)$ where $\delta'(q, s) = p$ such that $\delta(p, s) = q$

d) None of the above

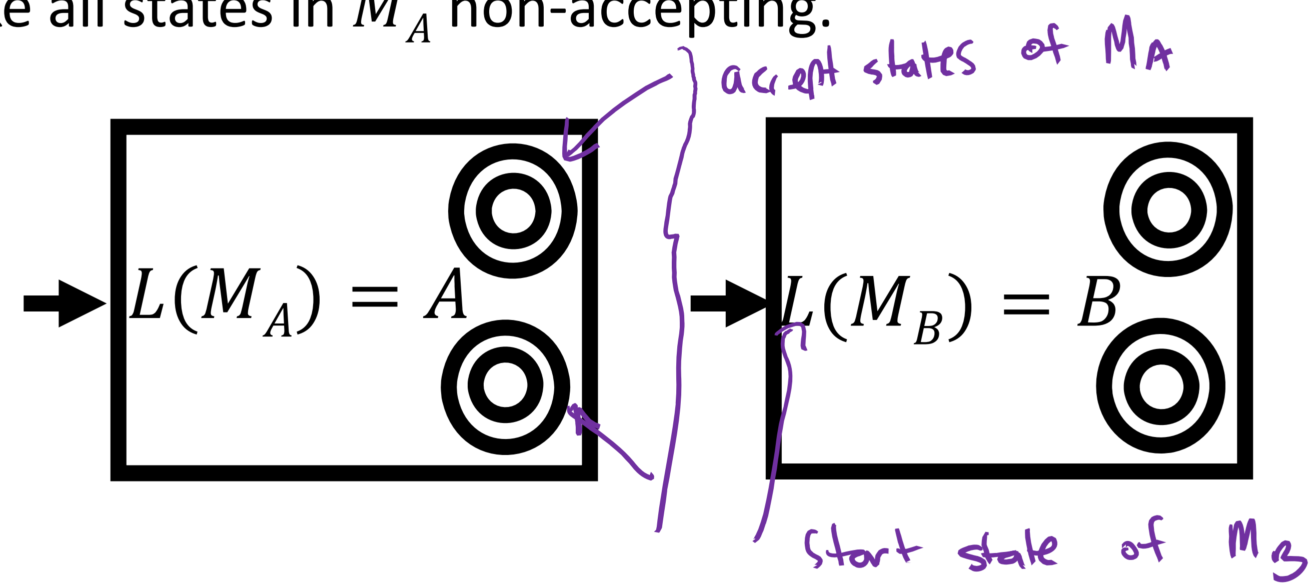
Closure under Concatenation

Concatenation: $A \circ B \; = \; \{ \, xy \; | x \in A, y \in B \, \}$

Theorem. If A and B are regular, then $A \circ B$ is also regular.

Proof idea: Given DFAs M_A and M_B, construct NFA by

- Connecting all accept states in M_A to the start state in M_B.
- Make all states in M_A non-accepting.



Closure under Concatenation

$= \{z \mid \exists x \in A, \exists y \in B,\ z = xy\}$

Concatenation: $A \circ B = \{ xy \mid x \in A, y \in B \}$

Theorem. If A and B are regular, then $A \circ B$ is also regular.

Proof idea: Given DFAs M_A and M_B, construct NFA by

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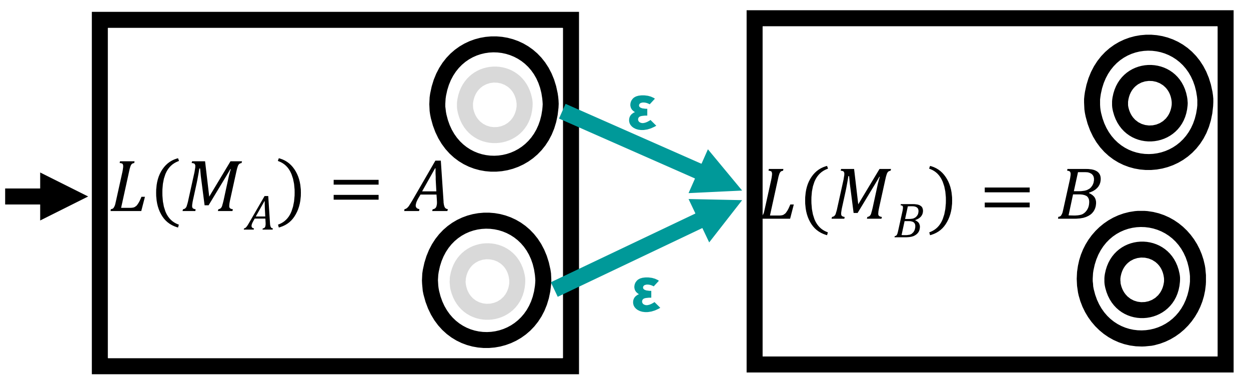
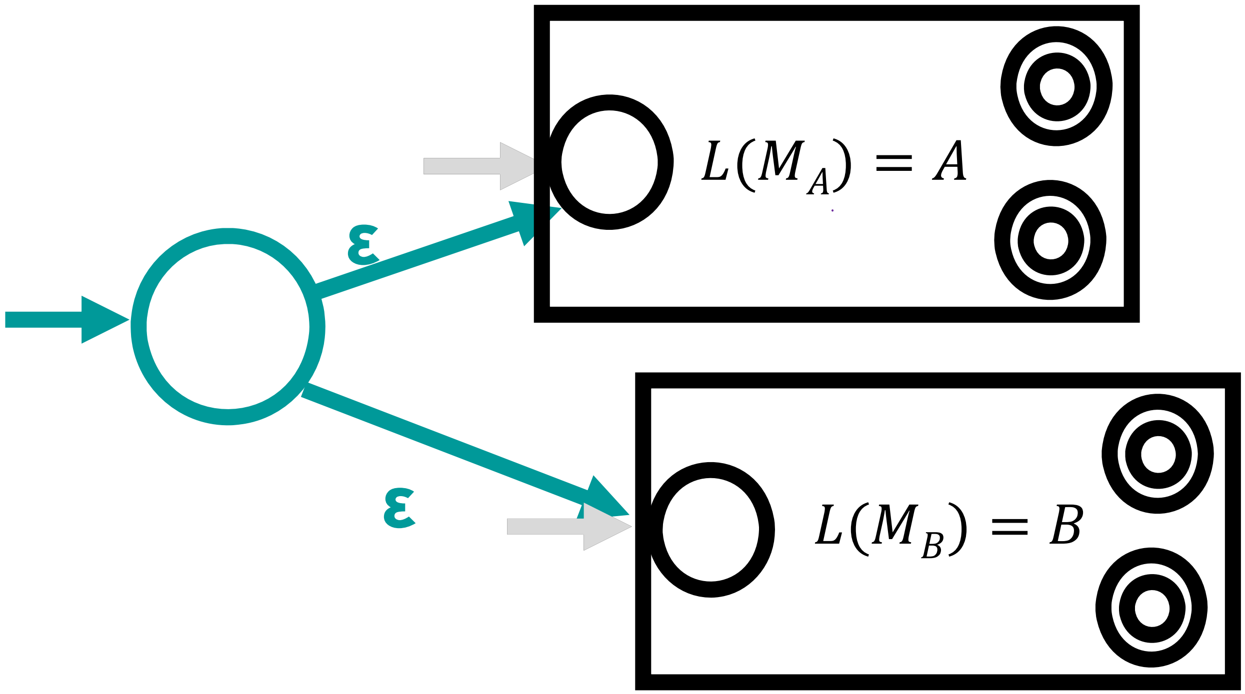


A Mystery Construction

Given DFAs M_A recognizing A and M_B recognizing B, what does the following NFA recognize?

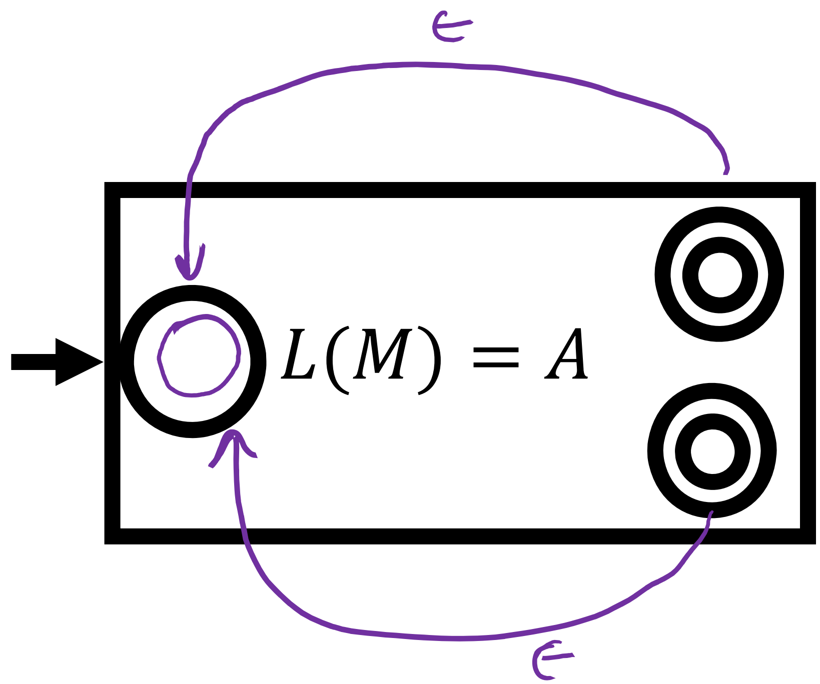


a) $A \cup B$
b) $A \circ B$
c) $A \cap B$
d) $\{\varepsilon\} \cup A \cup B$

Closure under Star

Star: $A^* = \{ a_1a_2...a_n | n \geq 0 \text{ and } a_i \in A\}$

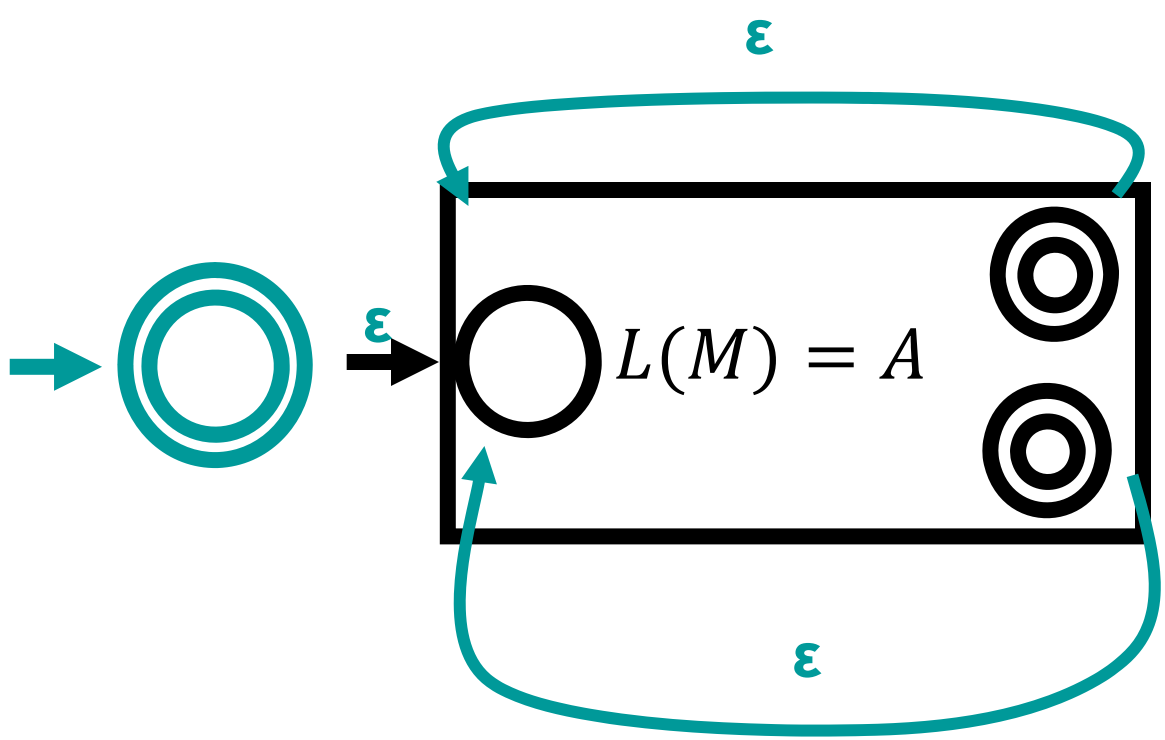
Theorem. If A is regular, then A^* is also regular.



Closure under Star

Star: $A^* = \{ a_1 a_2 \ldots a_n \mid n \geq 0 \text{ and } a_i \in A\}$

Theorem. If A is regular, then A^* is also regular.



On proving your own closure properties

You'll have homework/test problems of the form "show that the regular languages are closed under some operation"

What would Sipser do?

- Give the "proof idea": Explain how to take machine(s) recognizing regular language(s) and create a new machine
- Explain in a few sentences why the construction works
- Give a formal description of the construction
- No need to formally prove that the construction works

Regular Expressions

Regular Expressions

- A different way of describing regular languages
- A regular expression expresses a (possibly complex) language by combining simple languages using the regular operations

"Simple" languages: $\emptyset, \{\varepsilon\}, \{a\}$ for some $a \in \Sigma$

Regular operations:

Union: $A \cup B$

Concatenation: $A \circ B = \{ab \mid a \in A, b \in B\}$

Star: $A^* = \{ a_1 a_2 \ldots a_n \mid n \geq 0 \text{ and } a_i \in A\}$

Regular Expressions – Syntax

A regular expression R is defined recursively using the following rules:

1. ε, $\emptyset$, and a are regular expressions for every $a \in \Sigma$

2. If R_1 and R_2 are regular expressions, then so are $(R_1 \cup R_2)$, $(R_1 \circ R_2)$, and (R_1^*)

Examples: (over $\Sigma = \{a, b, c\}$)

$(a \circ b)$ $\quad$ $((((a \circ (b^*)) \circ c) \cup (((a^*) \circ b))^*))$ $\quad$ $(\emptyset^*)$

Regular Expressions – Semantics

$L(R)$ = the language a regular expression describes

1. $L(\emptyset) = \emptyset$
2. $L(\varepsilon) = \{\varepsilon\}$
3. $L(a) = \{a\}$ for every $a \in \Sigma$
4. $L((R_1 \cup R_2)) = L(R_1) \cup L(R_2)$
5. $L((R_1 \circ R_2)) = L(R_1) \circ L(R_2)$
6. $L\big((R_1^*)\big) = (L(R_1))^*$

Regular Expressions – Example

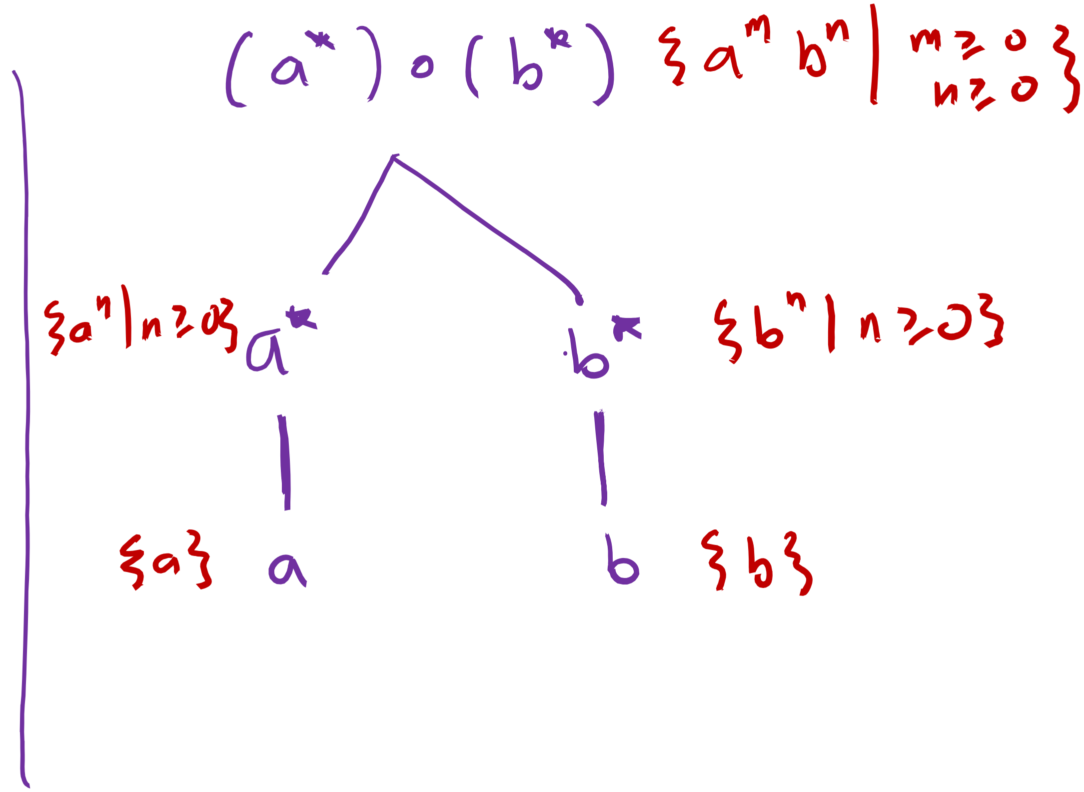
$L(((a^*) \circ (b^*))) =$

a) $\{a^n b^n \mid n \geq 0\}$

b) $\{a^m b^n \mid m, n \geq 0\}$

c) $\{(ab)^n \mid n \geq 0\}$

d) $\{a, b\}^*$



Simplifying Notation

- Omit $\circ$ symbol: $(ab) = (a \circ b)$

- Omit many parentheses, since union and concatenation are associative:

$$(a \cup b \cup c) = (a \cup (b \cup c)) = ((a \cup b) \cup c)$$

- Order of operations: Evaluate star, then concatenation, then union

$$ab^* \cup c = (a(b^*)) \cup c$$

Examples

$(1(0\cup 1))^*$

Let $\Sigma = \{0, 1\}$

1. $\{w \mid w$ contains exactly one $1\}$ $= L(0^* 1 0^*)$

$= \{0\} \cup \{1\}$

$\Sigma_1^2 \circ \{0\} \circ \Sigma_1^*$ $\quad \{0,1\} = \Sigma_1 = L(0 \cup 1)$

2. $\{w \mid w$ has length at least 3 and its third symbol is $0\}$

$= L((0\cup 1)(0 \cup 1) 0 (0\cup 1)^*) = L(\Sigma_1 \Sigma_1 0 \Sigma_1^*)$

3. $\{w \mid$ every odd position of w is $1\}$ counting from left

1	0	1	0	0
1	2	3	4	5

$L((1(0\cup 1))^* (1 \cup \epsilon))$

$= L((1(0\cup 1))^* 1^*)$

Syntactic Sugar

- For alphabet Σ, the regex Σ represents $L(\Sigma) = \Sigma$

- For regex R, the regex $R^{+} = RR^{*}$

(one or more copies from $L(R)$)

Regexes in the Real World

`grep` = globally search for a regular expression and print matching lines

```
$ grep '^xy*z' myfile
xyz
xyzde
xzz
xz
xyyz
xyyyz
xyyyyz
$ grep '^x.*z' myfile
xyz
xyzde
xxz
xzz
x\z
x*z
xz
x z
xYz
xyyz
xyyyz
xyyyyz
$ grep '^x\*z' myfile
x*z
$ grep '\\' myfile
x\z
$ 
```