BU CS 332 – Theory of Computation

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Lecture 11:

- TM Variants
- Nondeterministic TMs
- Church-Turing Thesis

Reading:

Sipser Ch 3.2

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Last Time

Formal definition of a TM, configurations, how a TM computes

Recognizability vs. Decidability:

A is Turing-recognizable if there exists a TM M such that

- $w \in A \implies M$ halts on w in state q_{accept}
- $w \notin A \implies M$ halts on w in state q_{reject} OR
  M runs forever on w

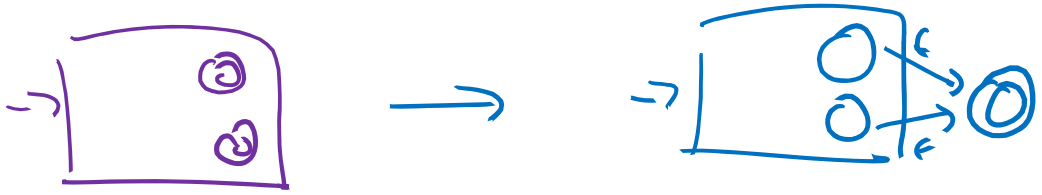
A is (Turing-)decidable if there exists a TM M such that

- $w \in A \implies M$ halts on w in state q_{accept}
- $w \notin A \implies M$ halts on w in state q_{reject}

How Robust is the TM Model?

Does changing the model result in different languages being recognizable / decidable?

So far we've seen…



- We can require that NFAs have a single accept state
- Adding nondeterminism does not change the languages recognized by finite automata

Other modifications possible too: E.g., allowing DFAs to have multiple passes over their input does not increase their power

Turing machines have an astonishing level of robustness

TMs are equivalent to…

- TMs with “stay put”
- TMs with 2-way infinite tapes
- Multi-tape TMs
- Nondeterministic TMs
- Random access TMs
- Enumerators
- Finite automata with access to an unbounded queue
- Primitive recursive functions
- Cellular automata

…

Equivalent TM models

- TMs that are allowed to "stay put" instead of moving left or right

$$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R, S\}$$

TMs with stay put are *at least* as powerful as basic TMs

(Every basic TM is a TM with stay put that never stays put)

How would you show that TMs with stay put are *no more* powerful than basic TMs?

a) Convert any basic TM into an equivalent TM with stay put

b) Convert any TM with stay put into an equivalent basic TM

c) Construct a language that is recognizable by a TM with stay put, but not by any basic TM

d) Construct a language that is recognizable by a basic TM, but not by any TM with stay put

Equivalent TM models

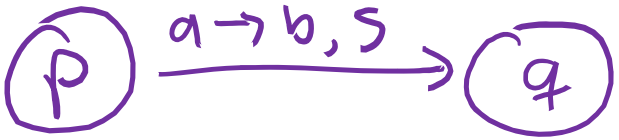
- TMs that are allowed to "stay put" instead of moving left or right

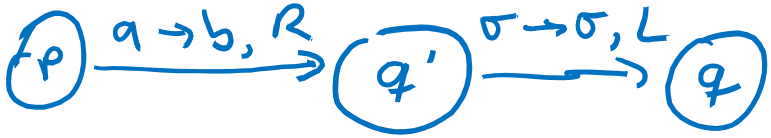
$$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R, S\}$$

Proof that TMs with stay put are no more powerful:

Simulation: Convert any TM M with stay put into an equivalent basic TM M'

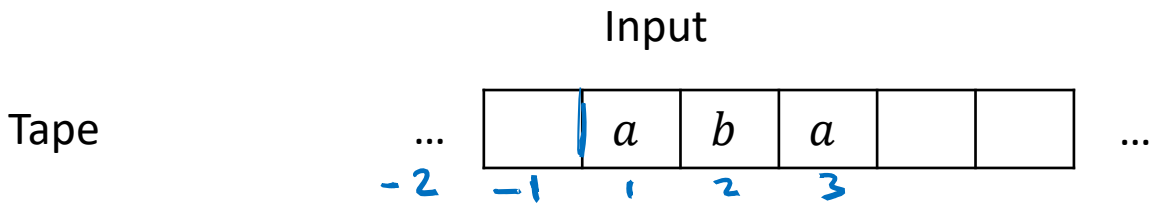
Replace every stay put instruction in M with a move right instruction, followed by a move left instruction in M'





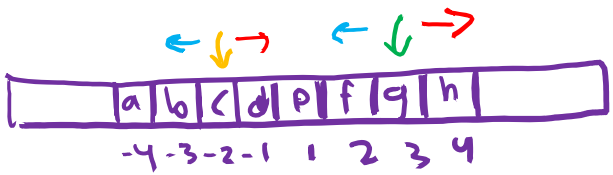
Equivalent TM models

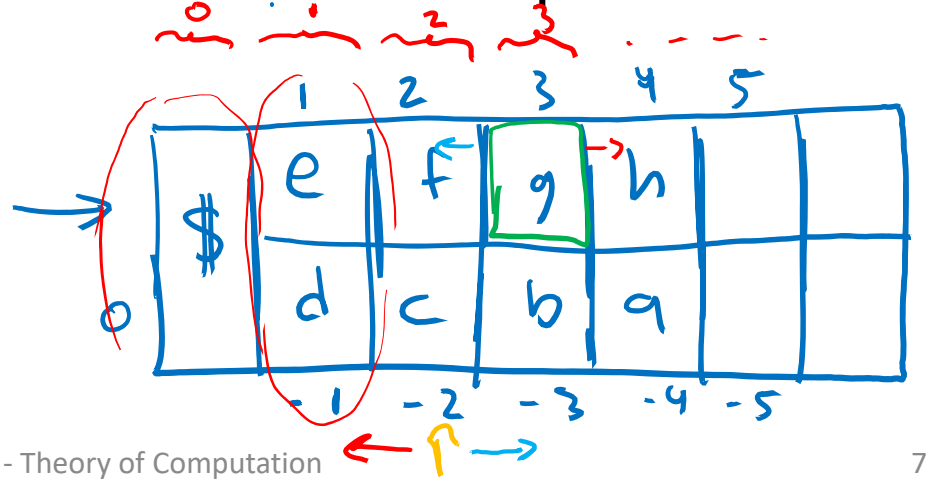
- TMs with a 2-way infinite tape, unbounded left to right



Proof that TMs with 2-way infinite tapes are no more powerful:

Simulation: Convert any TM M with 2-way infinite tape into a 1-way infinite TM M' with a "two-track tape"





Implementation-Level Simulation

Given 2-way TM M construct a basic TM M' as follows.

TM M' = "On input $w = w_1 w_2 \ldots w_n$:

1. Format 2-track tape with contents

$$\$, (w_1, \sqcup), (w_2, \sqcup), \ldots, (w_n, \sqcup)$$

top bottom

2. To simulate one move of M:

a) If working on upper track, read/write to the first position of cell under tape head, and move in the same direction as M

b) If working on lower track, read/write to second position of cell under tape head, and move in the opposite direction as M

c) If move results in hitting $, switch to the other track. "

Formalizing the Simulation

Given 2-way TM $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$, construct

$M' = (Q', \Sigma, \Gamma', \delta', q_0', q_{\text{accept}}', q_{\text{reject}}')$

top track

bottom track

New tape alphabet: $\Gamma' = (\Gamma \times \Gamma) \cup \{\$\}$

New state set: $Q' = Q \times \{+, -\}$

lower

$(q, -)$ means "q, working on ~~upper~~ track"

$(q, +)$ means "q, working on ~~lower~~ track"

upper

New transitions:

If $\delta(p, a_-) = (q, b, L)$, let $\delta'\big((p, -), (a_-, a_+)\big) = ((q, -), (b, a_+), R)$

Also need new transitions for moving right, lower track, hitting \$, initializing input into 2-track format

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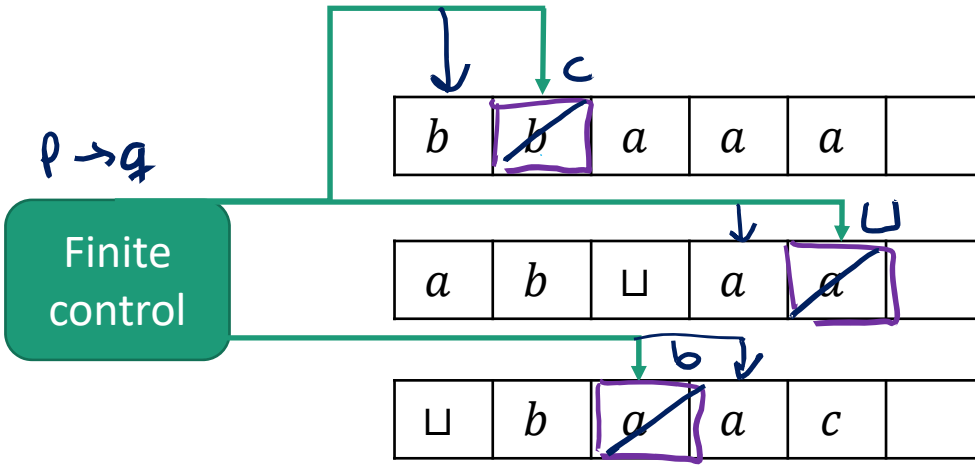
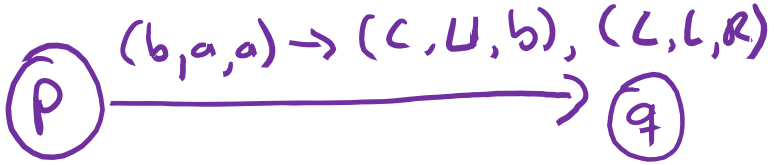
Multi-Tape TMs





Fixed number of tapes k

(k can't depend on input or change during computation)

Transition function $\delta : Q \times \Gamma^k \rightarrow Q \times \Gamma^k \times \{L, R, S\}^k$

current state; read k symbols; new state; write k new symbols; k movement instructions

Why are Multi-Tape TMs Helpful?

To show a language is Turing-recognizable or decidable, it's enough to construct a multi-tape TM

Often easier to construct multi-tape TMs

Ex. Decider for $\{a^i b^j | i > j\}$

On input w:

1) Scan tape 1 left-to-right to check that $w \in L(a^* b^*)$
2) Scan tape 2 left-to-right to copy all b's to tape 2
3) Starting from left ends of tapes 1 and 2, scan both tapes to check that every b on tape 2 has an accompanying a on tape 1. If not, reject.
4) Check that the first blank on tape 2 has an accompanying a on tape 1. If so, accept; otherwise, reject.

Why are Multi-Tape TMs Helpful?

To show a language is Turing-recognizable or decidable, it's enough to construct a multi-tape TM

Very helpful for proving closure properties

Ex. Closure of recognizable languages under union. Suppose M_1 is a single-tape TM recognizing L_1, M_2 is a single-tape TM recognizing L_2

Goal: Construct new TM M recognizing $L_1 \cup L_2$.

On input w:

1) Copy w to tape 2
2) Run M_1 on tape 1
   Run M_2 on tape 2
   If either accepts, accept. If both reject, reject.

If M_1 and M_2 are deciders:

$w \in L_1 \cup L_2 \Rightarrow$ either $M_1(w)$ accepts or $M_2(w)$ accepts and both halt.
$\Rightarrow M(w)$ accepts.

$w \notin L_1 \cup L_2 \Rightarrow$ both reject $\Rightarrow M(w)$ rejects

Why are Multi-Tape TMs Helpful?

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Very helpful for proving closure properties

Ex. Closure of recognizable languages under union. Suppose M_1 is a single-tape TM recognizing L_1, M_2 is a single-tape TM recognizing L_2

On input w:

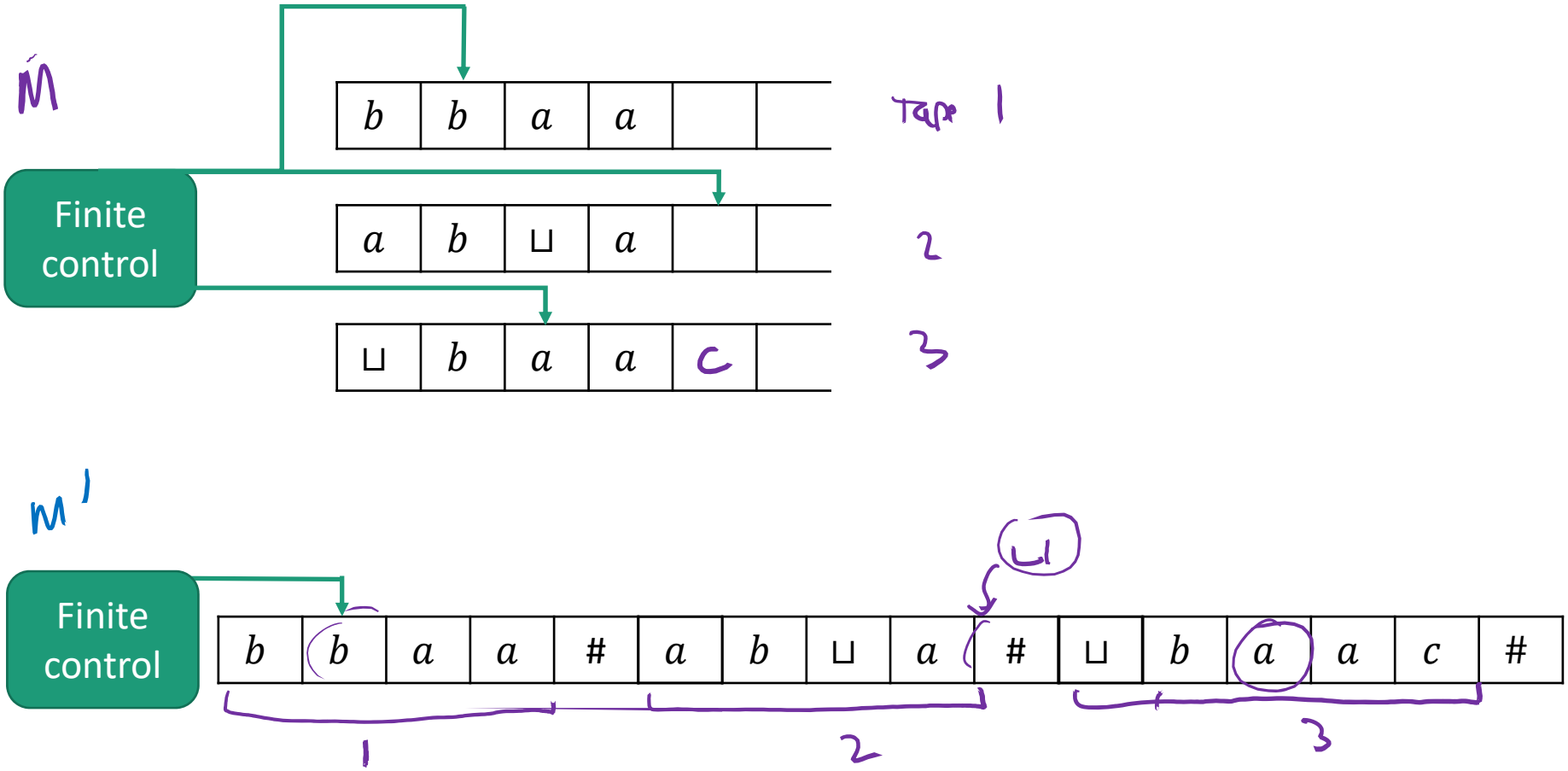
1) Scan tapes 1, 2, and 3 left-to-right to copy w to tapes 2 and 3
2) Repeat forever:
    a) Run M_1 for one step on tape 2
    b) Run M_2 for one step on tape 3
    c) If either machine accepts, accept

If $w \in L_1 \cup L_2$, then either M_1 or M_2 accepts

If $w \notin L_1 \cup L_2$, then neither M_1 nor M_2 will accept

Multi-Tape TMs are Equivalent to Single-Tape TMs

Theorem: Every k-tape TM M with can be simulated by an equivalent single-tape TM M'



How to Simulate It

To show that a TM variant is no more powerful than the basic, single-tape TM:

Show that if M is any variant machine, there exists a basic, single-tape TM M' that can simulate M

(Usual) parts of the simulation:

- Describe how to initialize the tapes of M' based on the input to M
- Describe how to simulate one step of M's computation using (possibly many steps of) M'

Simulating Multiple Tapes

Implementation-Level Description of M'

On input $w = w_1 w_2 \ldots w_n$

1. Format tape into $\#\, \dot{w}_1 w_2 \ldots w_n \#\, \dot{\sqcup}\, \#\, \dot{\sqcup}\, \#\, \ldots \#$
2. For each move of M:

   Scan left-to-right, finding current symbols

   Scan left-to-right, writing new symbols,

   Scan left-to-right, moving each tape head

   If a tape head goes off the right end, insert blank

   If a tape head goes off left end, move back right

Closure Properties

The Turing-decidable languages are closed under:

- Union
- Concatenation
- Star
- Intersection
- Reverse
- Complement

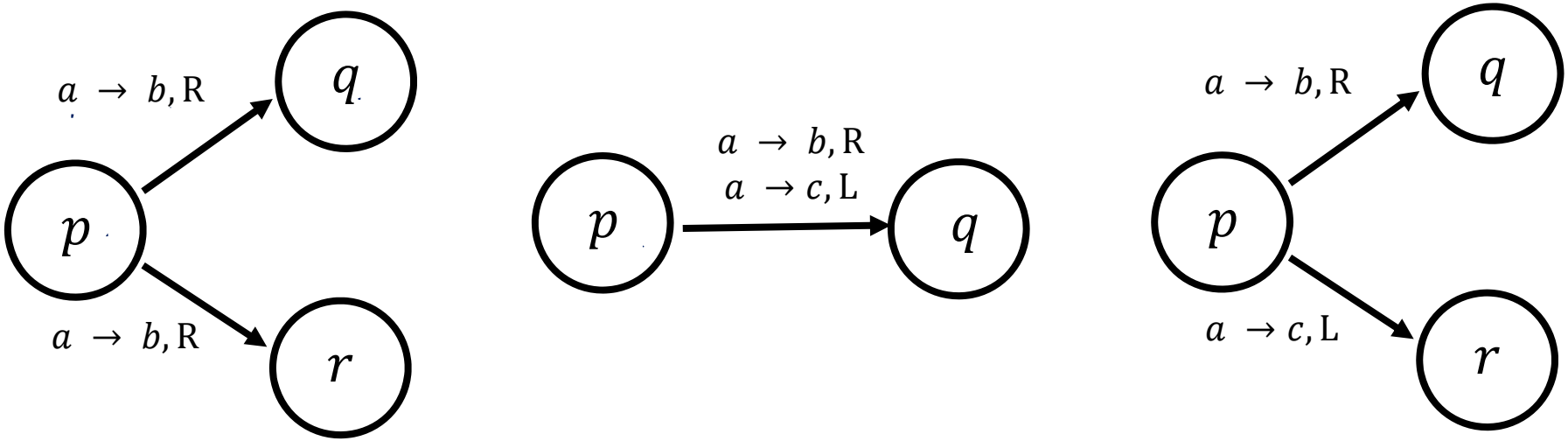
The Turing-recognizable languages are closed under:

- Union
- Concatenation
- Star
- Intersection
- Reverse

Nondeterministic TMs

At any point in computation, may nondeterministically branch. Accepts iff there exists an accepting branch.

Transition function $\delta : Q \times \Gamma \rightarrow P(Q \times \Gamma \times \{\mathrm{L}, \mathrm{R}, \mathrm{S}\})$



Nondeterministic TMs

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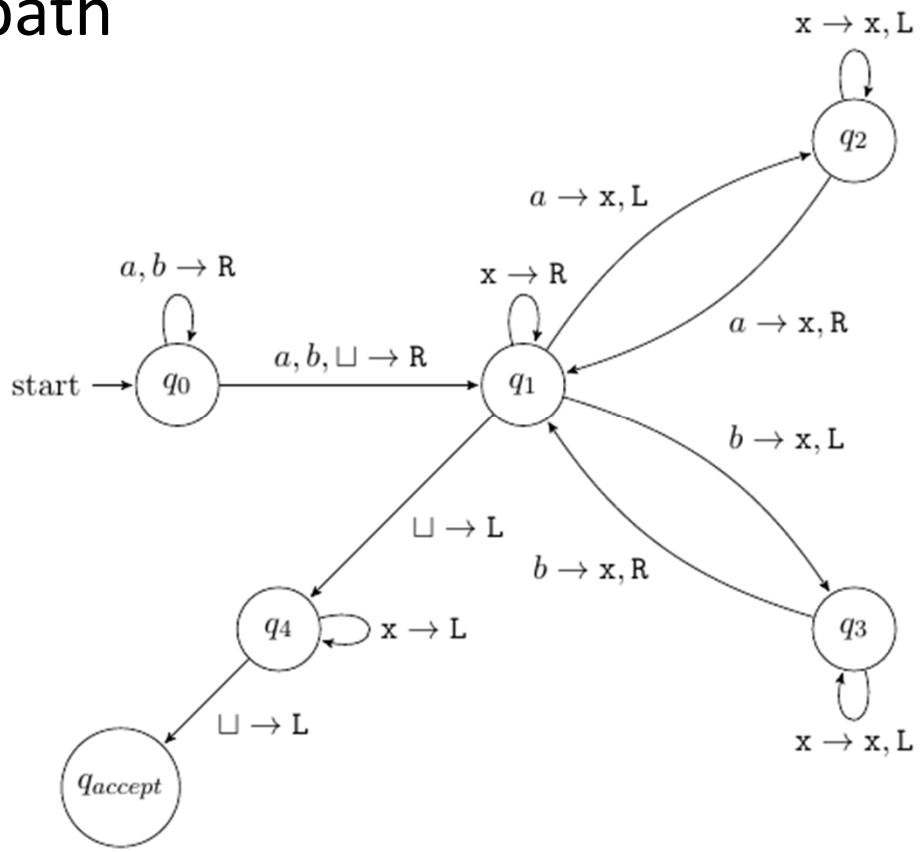

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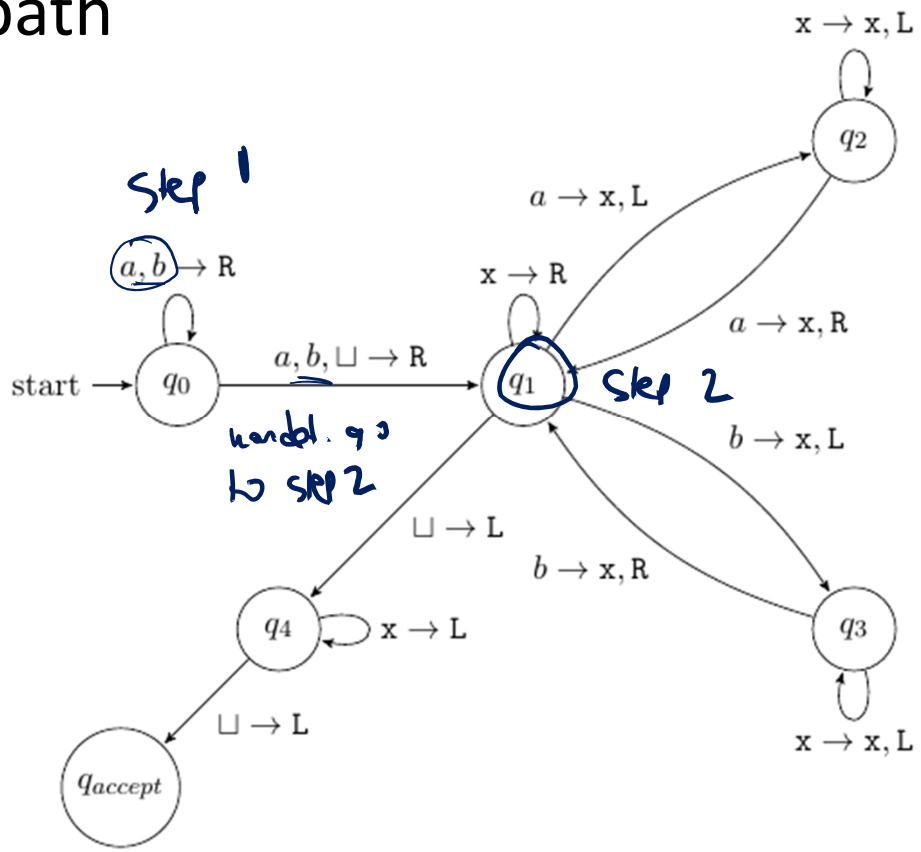

What is the language recognized by this NTM?

a) $\{ww \mid w \in \{a,b\}^*\}$
b) $\{ww^R \mid w \in \{a,b\}^*\}$
c) $\{ww \mid w \in \{a,b,\mathrm{x}\}^*\}$
d) $\{w\mathrm{x}^n w^R \mid w \in \{a,b\}^*, n \geq 0\}$

Nondeterministic TMs

At any point in computation, may nondeterministically branch. Accepts iff there exists an accepting computation path

Implementation-Level Description

On input string w:

1) Scan tape left-to-right. At some point, nondeterministically go to step 2
2) a) Read the next symbol s and cross it off
   b) Move the head left repeatedly until a non-X symbols is found. If it matches s, cross it off. Else, reject.
   c) Move the head right until a non-X symbol is found. If blank is hit, go to step 3.
   d) Go back to 2a)
3) Check that the entire tape consists of X's. If so, accept. Else, reject.