BU CS 332 – Theory of Computation

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Lecture 12:

- Nondeterministic TMs
- Church-Turing Thesis
- Decidable Problems

Reading:

Sipser Ch 3.2, 4.1

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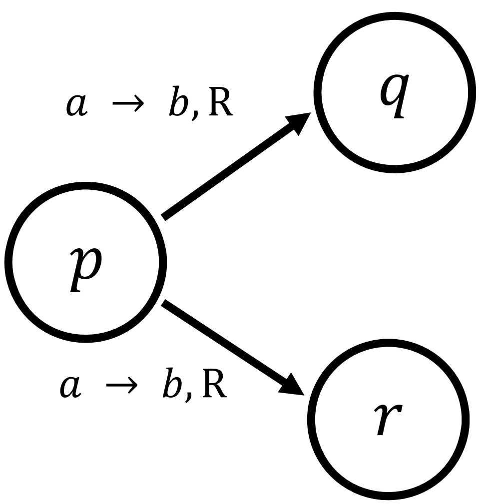
October 20, 2022

Nondeterministic TMs

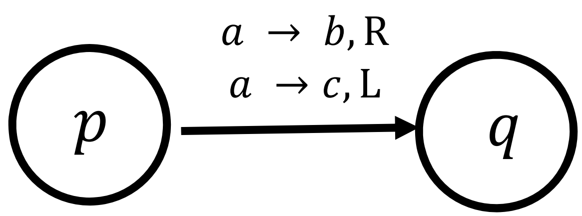
At any point in computation, may nondeterministically branch. Accepts iff there exists an accepting branch.

Transition function $\delta : Q \times \Gamma \rightarrow P(Q \times \Gamma \times \{\mathrm{L}, \mathrm{R}, S\})$

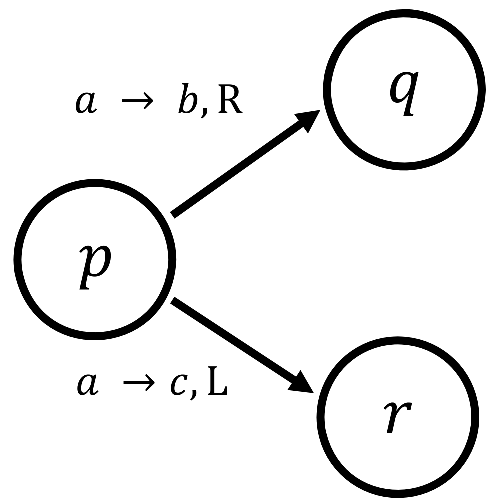
Transition function can lead to multiple states



…or give multiple write/movement instructions



…or both



Nondeterministic TMs

Input abba

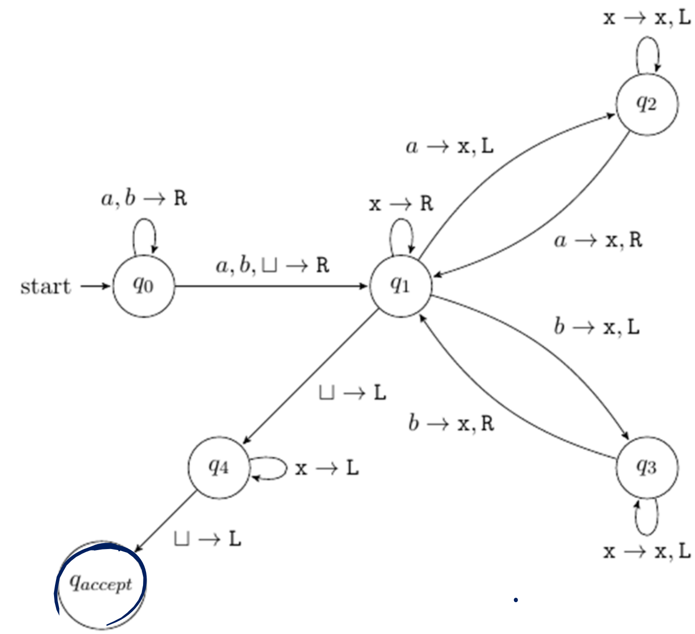
q_0 abba
a q_0 bba
ab q_0 ba
abb q_0 a
abba q_0 ⊔
abba⊔ q_1 ⊔

One branch

abba q_4 ⊔
q_4 abba
q_{reject} abba

Another branch

q_0 abba
a q_0 bba
ab q_1 ba
~~abba~~
q_{accept} x x x x



On input string w:

1) Scan tape left-to-right. At some point, nondeterministically go to step 2
2) a) Read the next symbol s and cross it off
   b) Move the head left repeatedly until a non-x symbol is found. If it matches s, cross it off. Else, reject.
   c) Move the head right until a non-x symbol is found. If blank is hit, go to step 3.
   d) Go back to 2a)
3) Check that the entire tape consists of x's. If so, accept. Else, reject.

Language of this NTM = $\{ w w^R \mid w \in \{a,b\}^* \}$

Nondeterministic TMs

Ex. Given TMs M_1 and M_2, construct an NTM recognizing $L(M_1) \cup L(M_2)$

On input w:

1) Nondeterministically either:

   a) Run M_1 on w. Accept if it accepts

   b) Run M_2 on w. Accept if it accepts.

NTM N

$w \in L(M_1) \cup L(M_2)$. Then either M_1 or M_2 accepts w.

If M_1 accepts w, branch (a) leads N to accept w

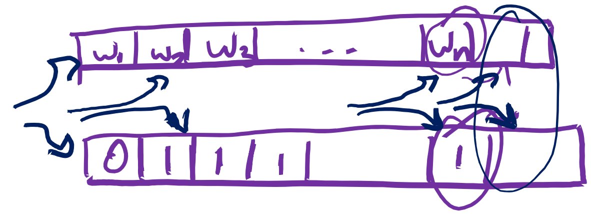
If M_2 accepts w, branch (b) leads N to accept w.

$\Rightarrow$ N accepts w.

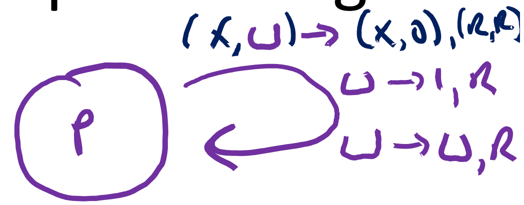
$w \notin L(M_1) \cup L(M_2)$. Then no branch of computation leads N to accept

$\Rightarrow$ N does not accept w.

Nondeterministic TMs



Ex. NTM for $L = \{w \mid w \text{ is a binary number representing the product of two integers } a, b \geq 2\}$



High-Level Description:

On input w:

1) Nondeterministically guess $a \in \{2, \ldots, w\}$, $b \in \{2, \ldots, w\}$
2) Multiply $a \times b$, accept if $= w$.

Correctness analysis: (In our heads)

If $w \in L$: then $\exists a, b \in \{2, \ldots, w\}$ s.t. $a \times b = w$.
$\Rightarrow$ The branch of computation where a, b guessed lead NTM to accept.

If $w \notin L$: No guess of a, b will cause $a \times b = w$
$\Rightarrow$ NTM does not accept.

Nondeterministic TMs

An NTM N accepts input w if when run on w it accepts on at least one computational branch

$$L(N) = \{w \mid N \text{ accepts input } w\}$$

N recognizes L means:

- If $w \in L$, there exists a computational branch of N on input w that leads it to accept
- If $w \notin L$, then every branch of N on w either rejects or loops forever.

An NTM N is a decider if on **every** input, it halts on **every** computational branch

N decides L means:

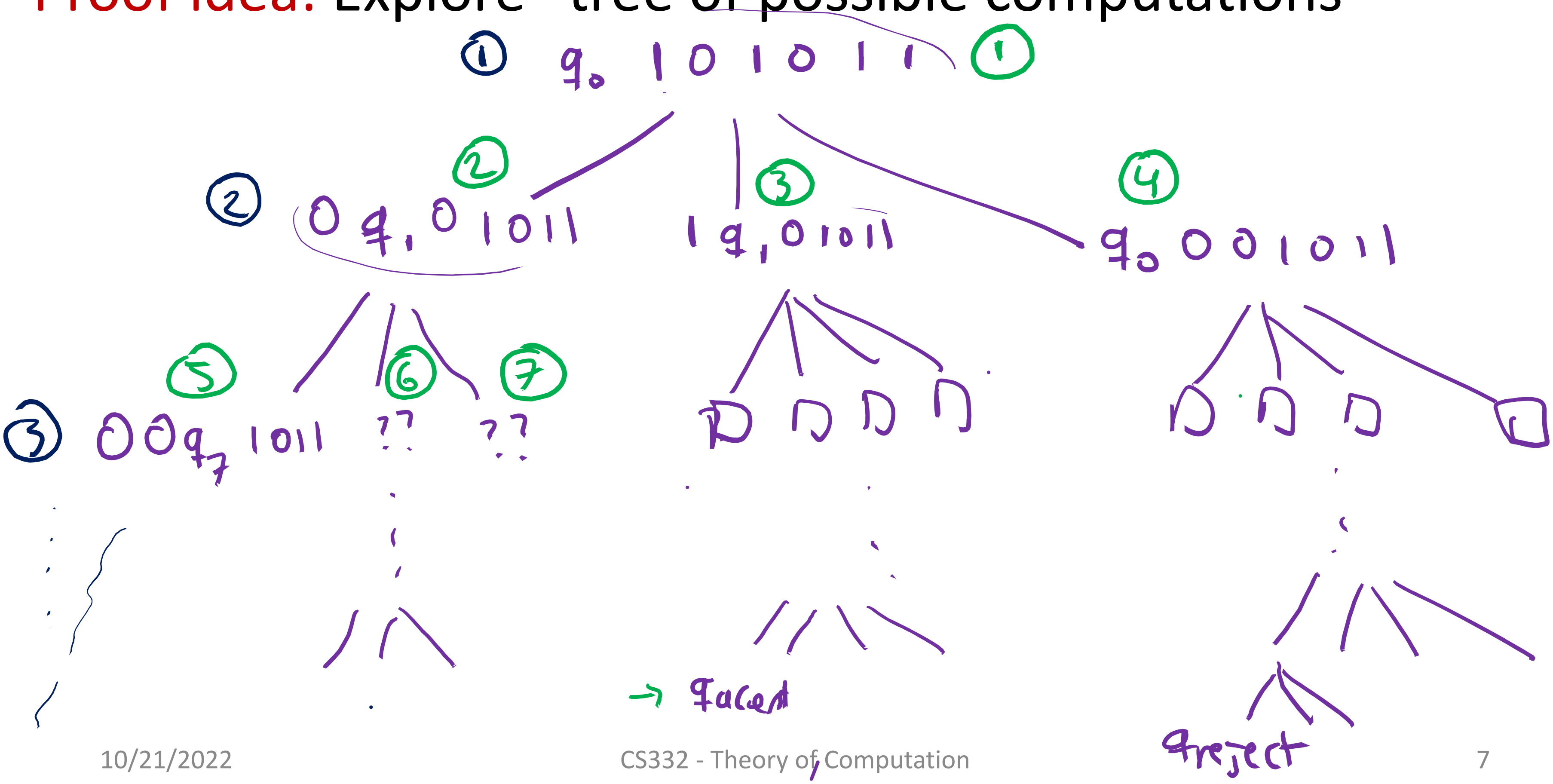
- If $w \in L$, there exists an accepting branch
- If $w \notin L$, every branch leads to reject.

Nondeterministic TMs

1 DFS
2 BFS

Theorem: Every nondeterministic TM can be simulated by an equivalent deterministic TM

Proof idea: Explore "tree of possible computations"



Simulating NTMs

Which of the following algorithms is always appropriate for searching the tree of possible computations for an accepting configuration?

a) Depth-first search: Explore as far as possible down each branch before backtracking

Works if NTM is a decider

b) Breadth-first search: Explore all configurations at depth 1, then all configurations at depth 2, etc.

Always works, even if NTM is a recognizer

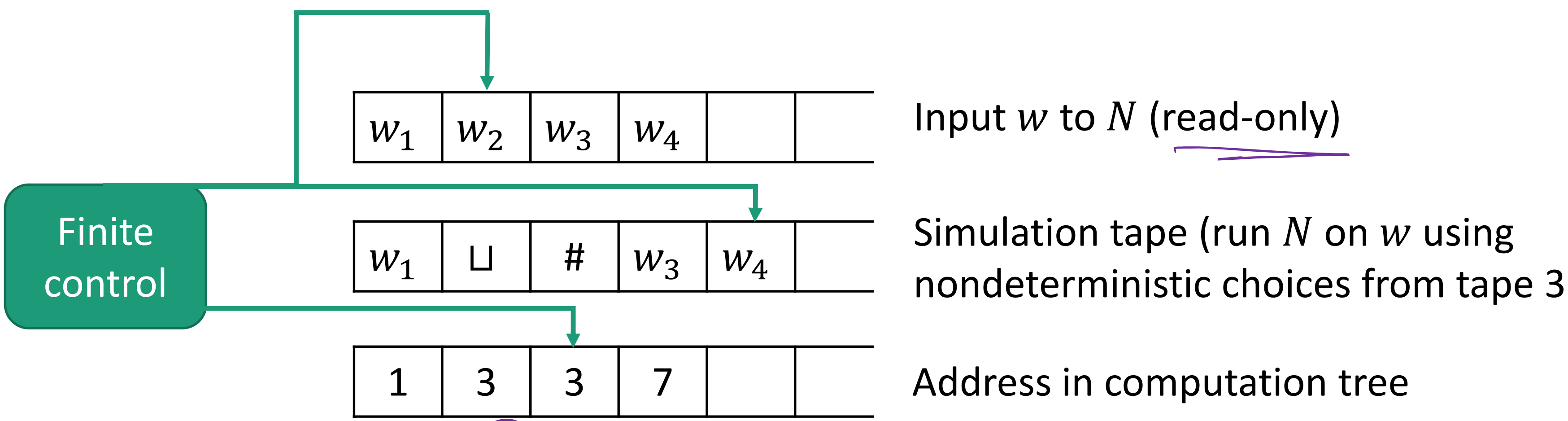
c) Both algorithms will always work

Nondeterministic TMs

Theorem: Every nondeterministic TM has an equivalent deterministic TM

Proof idea: Simulate an NTM N using a 3-tape TM

(See Sipser for full description)



3rd choice @ level 2

1st nondet. choice @ level 1

TMs are equivalent to…

- TMs with “stay put”
- TMs with 2-way infinite tapes
- Multi-tape TMs
- Nondeterministic TMs
- Random access TMs
- Enumerators
- Finite automata with access to an unbounded queue
- Primitive recursive functions
- Cellular automata

…

Church-Turing Thesis

The equivalence of these models is a mathematical theorem (you can prove that each can simulate another)

Church-Turing Thesis v1: The basic TM (hence all of these models) captures our intuitive notion of algorithms

prescriptive, normative

Church-Turing Thesis v2: Any physically realizable model of computation can be simulated by the basic TM

empirical

The Church-Turing Thesis is not a mathematical statement! Can't be mathematically proved

Decidable Languages

1928 – The *Entscheidungsproblem*

The "Decision Problem"

mathematical statement

TM

Is there an algorithm which takes as input a formula (in first-order logic) and decides whether it's logically valid?

is it true or false?

Questions

- Can we automate the job of mathematicians?
- Can we automate the tasks we did in the first part of the course on regular languages?

Questions about regular languages

- Given a DFA D and a string w, does D accept input w?
- Given a DFA D, does D recognize the empty language?
- Given DFAs D_1, D_2, do they recognize the same language?

(Same questions apply to NFAs, regexes)

Goal: Formulate each of these questions as a language, and decide them using Turing machines

Questions about regular languages

Design a TM which takes as input a DFA D and a string w, and determines whether D accepts w

How should the input to this TM be represented?

Let $D = (Q, \Sigma, \delta, q_0, F)$. List each component of the tuple separated by #

- Represent Q by ,-separated binary strings
- Represent Σ by ,-separated binary strings
- Represent $\delta : Q \times \Sigma \to Q$ by a ,-separated list of triples (p, a, q), …

"to String ()"

Denote the encoding of D, w by $\langle D, w \rangle$

Representation independence

Computability (i.e., decidability and recognizability) is **not** affected by the precise choice of encoding

Encodings $[\cdot]$, $\langle \cdot \rangle$

$\exists$ a TM M that, on input $[D]$ outputs $\langle D \rangle$

Why? A TM can always convert between different (reasonable) encodings

Given a TM that recognizes $L = \{ \langle D \rangle \mid D \ldots \}$

Construct a TM that recognizes $L' = \{ [D] \mid D \ldots \}$

On input $[D]$:

1) Run $M([D])$, producing $\langle D \rangle$
2) Run recognizer for L on $\langle D \rangle$.

From now on, we'll take ⟨ ⟩ to mean "any reasonable encoding"

A "universal" algorithm for recognizing regular languages

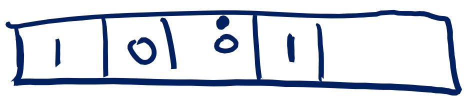
Computational problem:

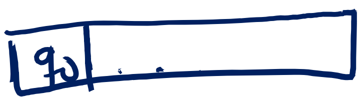
Given DFA D, string w, does D accept an input w?

$A_{\text{DFA}} = \{\langle D, w\rangle \mid \text{DFA } D \text{ accepts } w\}$

Theorem: A_{DFA} is decidable

Proof: Define a (high-level) 3-tape TM M on input $\langle D, w\rangle$:

1. Check if $\langle D, w\rangle$ is a valid encoding (reject if not) OK to omit
2. Simulate D on w, i.e.,
   - Tape 2: Maintain w and head location of D 

   - Tape 3: Maintain state of D, update according to δ 

3. Accept if D ends in an accept state, reject otherwise