BU CS 332 – Theory of Computation

https://forms.gle/bAZkPdxJAgoinYfm9

Lecture 12:

- Nondeterministic TMs
- Church-Turing Thesis
- Decidable Problems

Reading:

Sipser Ch 3.2, 4.1

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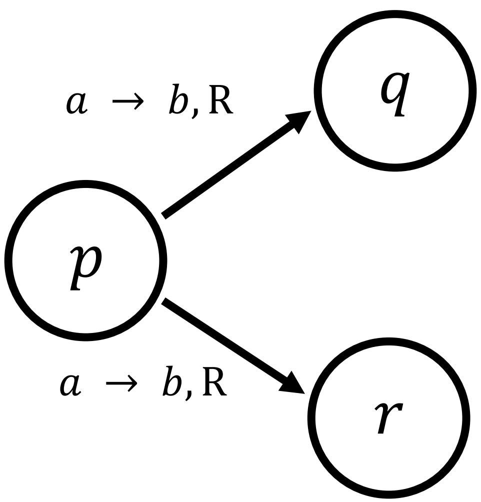
Nondeterministic TMs

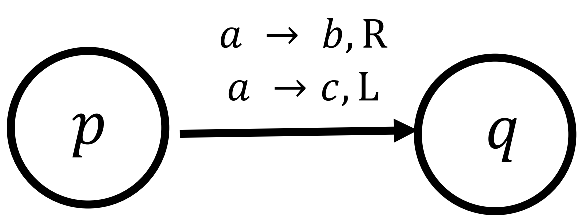
At any point in computation, may nondeterministically branch. Accepts iff there exists an accepting branch.

Transition function $\delta : Q \times \Gamma \rightarrow P(Q \times \Gamma \times \{\mathrm{L}, \mathrm{R}, \mathrm{S}\})$

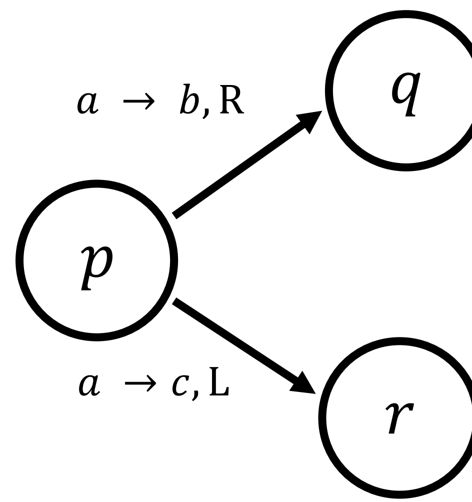
Transition function can lead to multiple states



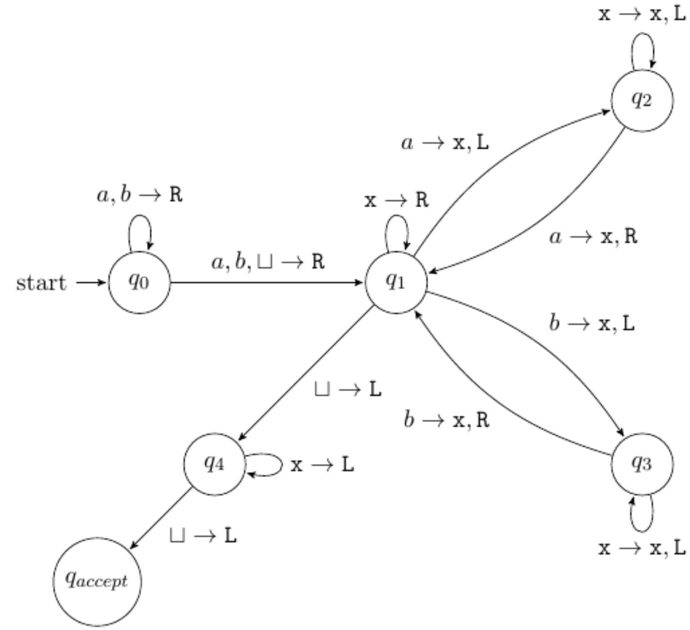
...or give multiple write/movement instructions



...or both



Nondeterministic TMs



On input string w:

1) Scan tape left-to-right. At some point, nondeterministically go to step 2
2) a) Read the next symbol s and cross it off
   b) Move the head left repeatedly until a non-x symbol is found. If it matches s, cross it off. Else, reject.
   c) Move the head right until a non-x symbol is found. If blank is hit, go to step 3.
   d) Go back to 2a)
3) Check that the entire tape consists of x's. If so, accept. Else, reject.

Nondeterministic TMs

Ex. Given TMs M_1 and M_2, construct an NTM recognizing $L(M_1) \cup L(M_2)$

Nondeterministic TMs

Ex. NTM for $L = \{w \mid w$ is a binary number representing the product of two integers $a, b \geq 2\}$

<u>High-Level Description:</u>

Nondeterministic TMs

An NTM N accepts input w if when run on w it accepts on at least one computational branch

$$L(N) = \{w \mid N \text{ accepts input } w\}$$

An NTM N is a decider if on **every** input, it halts on **every** computational branch

Nondeterministic TMs

Theorem: Every nondeterministic TM can be simulated by an equivalent deterministic TM

Proof idea: Explore "tree of possible computations"

Simulating NTMs

Which of the following algorithms is always appropriate for searching the tree of possible computations for an accepting configuration?

a) Depth-first search: Explore as far as possible down each branch before backtracking

b) Breadth-first search: Explore all configurations at depth 1, then all configurations at depth 2, etc.

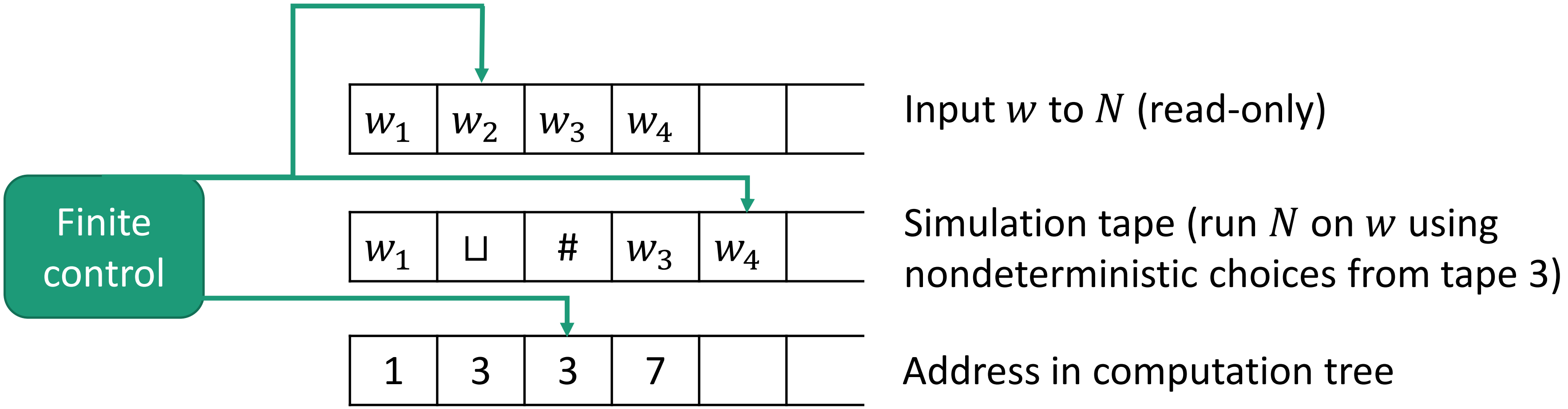
c) Both algorithms will always work

Nondeterministic TMs

Theorem: Every nondeterministic TM has an equivalent deterministic TM

Proof idea: Simulate an NTM N using a 3-tape TM

(See Sipser for full description)



TMs are equivalent to…

- TMs with “stay put”
- TMs with 2-way infinite tapes
- Multi-tape TMs
- Nondeterministic TMs
- Random access TMs
- Enumerators
- Finite automata with access to an unbounded queue
- Primitive recursive functions
- Cellular automata

…

Church-Turing Thesis

The equivalence of these models is a mathematical theorem (you can prove that each can simulate another)

Church-Turing Thesis v1: The basic TM (hence all of these models) captures our intuitive notion of algorithms

Church-Turing Thesis v2: Any physically realizable model of computation can be simulated by the basic TM

The Church-Turing Thesis is not a mathematical statement! Can't be mathematically proved

Decidable Languages

1928 – The *Entscheidungsproblem*

The "Decision Problem"

Is there an algorithm which takes as input a formula (in first-order logic) and decides whether it's logically valid?

Questions about regular languages

- Given a DFA D and a string w, does D accept input w?
- Given a DFA D, does D recognize the empty language?
- Given DFAs D_1, D_2, do they recognize the same language?

(Same questions apply to NFAs, regexes)

Goal: Formulate each of these questions as a language, and decide them using Turing machines

Questions about regular languages

Design a TM which takes as input a DFA D and a string w, and determines whether D accepts w

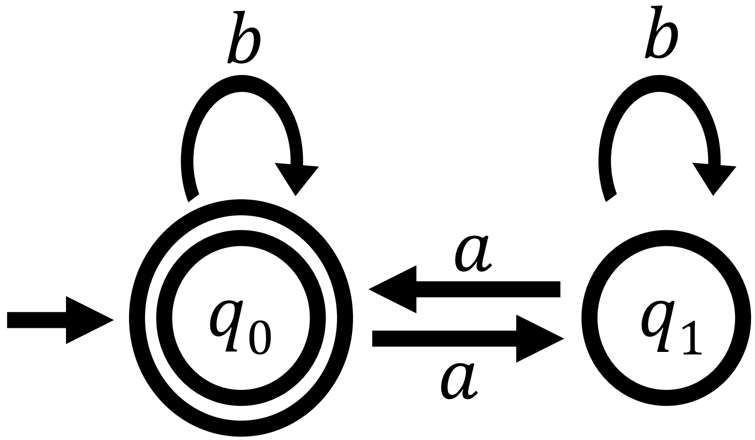
How should the input to this TM be represented?

Let $D = (Q, \Sigma, \delta, q_0, F)$. List each component of the tuple separated by #

- Represent Q by ,-separated binary strings
- Represent Σ by ,-separated binary strings
- Represent $\delta : Q \times \Sigma \to Q$ by a ,-separated list of triples (p, a, q), …

Denote the encoding of D, w by $\langle D, w \rangle$

Example



Representation independence

Computability (i.e., decidability and recognizability) is **not** affected by the precise choice of encoding

Why? A TM can always convert between different (reasonable) encodings

From now on, we'll take ⟨ ⟩ to mean "any reasonable encoding"

A "universal" algorithm for recognizing regular languages

$A_{\mathrm{DFA}} = \{\langle D, w\rangle \mid \text{DFA } D \text{ accepts } w\}$

Theorem: A_{DFA} is decidable

Proof: Define a (high-level) 3-tape TM M on input $\langle D, w\rangle$:

1. Check if $\langle D, w\rangle$ is a valid encoding (reject if not)
2. Simulate D on w, i.e.,
   - Tape 2: Maintain w and head location of D
   - Tape 3: Maintain state of D, update according to δ
3. Accept if D ends in an accept state, reject otherwise

Other decidable languages

$A_{\text{DFA}} = \{\langle D, w\rangle \mid \text{DFA } D \text{ accepts } w\}$

$A_{\text{NFA}} = \{\langle N, w\rangle \mid \text{NFA } N \text{ accepts } w\}$

$A_{\text{REX}} = \{\langle R, w\rangle \mid \text{regular expression } R \text{ generates } w\}$

NFA Acceptance

Which of the following describes a **decider** for $A_{\text{NFA}} = \{\langle N, w\rangle \mid \text{NFA } N \text{ accepts } w\}$?

a) Using a deterministic TM, simulate N on w, always making the first nondeterministic choice at each step. Accept if it accepts, and reject otherwise.

b) Using a deterministic TM, simulate all possible choices of N on w for 1 step of computation, 2 steps of computation, etc. Accept whenever some simulation accepts.

c) Use the subset construction to convert N to an equivalent DFA M. Simulate M on w, accept if it accepts, and reject otherwise.

Regular Languages are Decidable

Theorem: Every regular language L is decidable

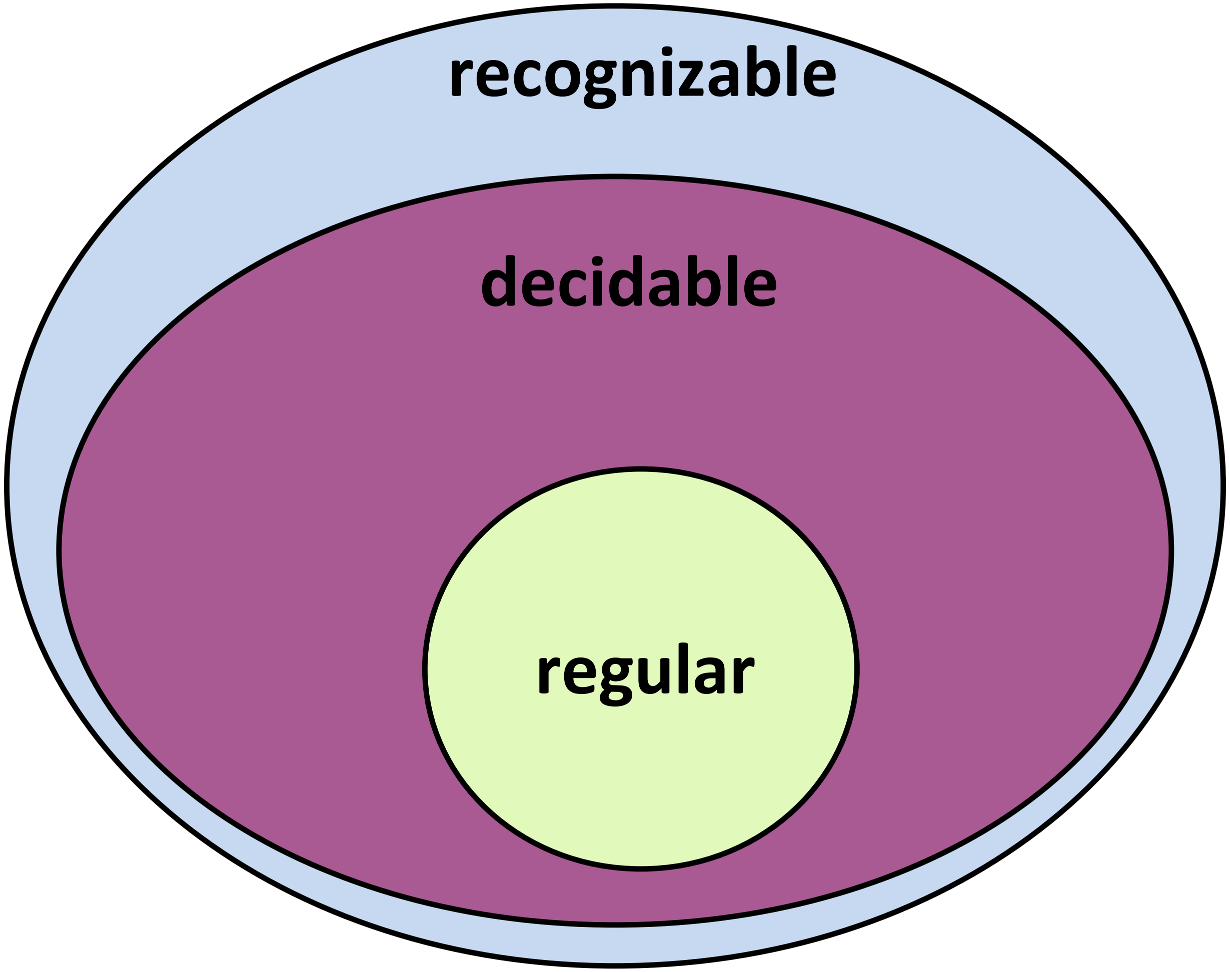
Proof 1: If L is regular, it is recognized by a DFA D. Convert this DFA to a TM M. Then M decides L.

Proof 2: If L is regular, it is recognized by a DFA D. The following TM M_D decides L.

On input w:

1. Run the decider for A_{DFA} on input $\langle D, w \rangle$
2. Accept if the decider accepts; reject otherwise

Classes of Languages



More Decidable Languages: Emptiness Testing

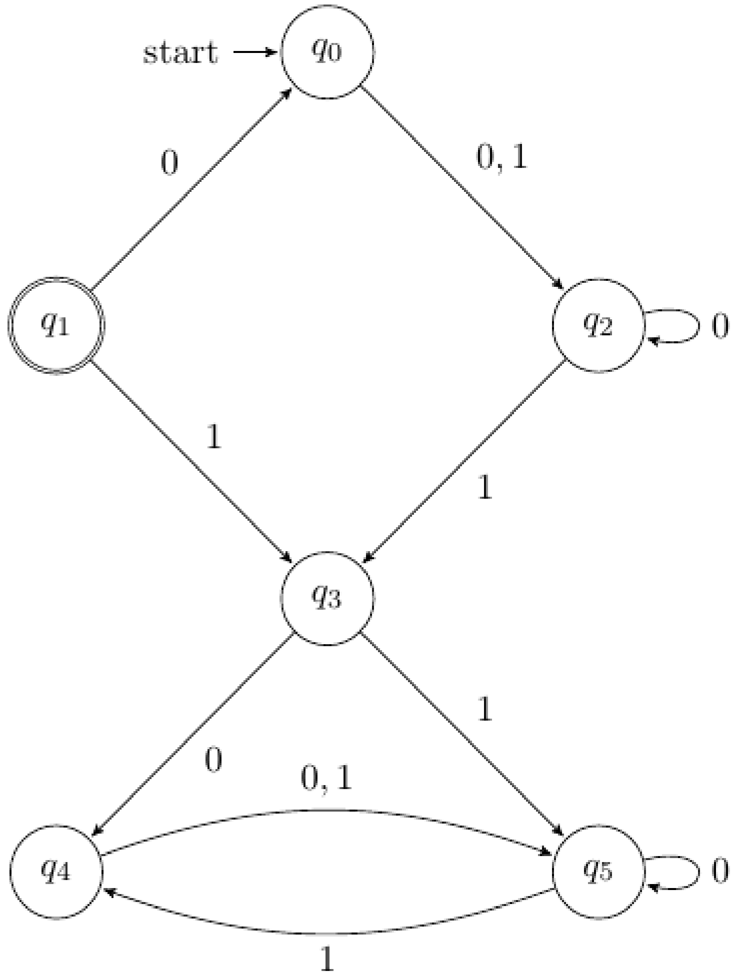
Theorem: $E_{\text{DFA}} = \{\langle D\rangle \mid D \text{ is a DFA such that } L(D) = \emptyset\}$ is decidable

Proof: The following TM decides E_{DFA}

On input $\langle D\rangle$, where D is a DFA with k states:

1. Perform k steps of breadth-first search on state diagram of D to determine if an accept state is reachable from the start state
2. Reject if a DFA accept state is reachable; accept otherwise

E_{DFA} Example



New Deciders from Old: Equality Testing

$EQ_{\mathrm{DFA}} = \{\langle D_1, D_2\rangle \mid D_1, D_2 \text{ are DFAs and } L(D_1) = L(D_2)\}$

Theorem: EQ_{DFA} is decidable

Proof: The following TM decides EQ_{DFA}

On input $\langle D_1, D_2\rangle$, where $\langle D_1, D_2\rangle$ are DFAs:

1. Construct DFA D recognizing the **symmetric difference** $L(D_1) \triangle L(D_2)$
2. Run the decider for E_{DFA} on $\langle D\rangle$ and return its output

Symmetric Difference

$$A \triangle B = \{w \mid w \in A \text{ or } w \in B \text{ but not both}\}$$