BU CS 332 – Theory of Computation

https://forms.gle/KujctosE3s84KLHX8

Lecture 14:

- Countability
- Uncountability / diagonalization
- Undecidable languages

Reading:

Sipser Ch 4.2

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Last Time

Universal Turing machine

A recognizer for $A_{\mathrm{TM}} = \{\langle M, w\rangle \mid \mathrm{TM}\ M \text{ accepts input } w\}$

…but not a decider

Today: Some languages, including A_{TM}, are *undecidable*

But first, a math interlude…

Countability and Diagonalization

How can we compare sizes of infinite sets?

Definition: Two sets have the same size if there is a bijection between them

A set is countable if

- it is a finite set, or
- it has the same size as $\mathbb{N}$, the set of natural numbers

Examples of countable sets

- $\emptyset$
- $\{0,1\}$
- $\{0, 1, 2, \ldots, 8675309\}$

- $E \ = \ \{2, 4, 6, 8, \ldots\}$
- $SQUARES = \{1, 4, 9, 16, 25, \ldots\}$
- $POW2 = \{2, 4, 8, 16, 32, \ldots\}$

$$|E| = |SQUARES| = |POW2| = |\mathbb{N}|$$

How to show that $\mathbb{N} \times \mathbb{N}$ is countable?

$$\begin{array}{cccccc}
(1,1) & (2,1) & (3,1) & (4,1) & (5,1) & \cdots \\
(1,2) & (2,2) & (3,2) & (4,2) & (5,2) & \cdots \\
(1,3) & (2,3) & (3,3) & (4,3) & (5,3) & \cdots \\
(1,4) & (2,4) & (3,4) & (4,4) & (5,4) & \cdots \\
(1,5) & (2,5) & (3,5) & (4,5) & (5,5) & \\
 & & & & & \ddots
\end{array}$$

How to argue that a set S is countable

- Describe how to list the elements of S, usually in stages:

Ex: Stage 1) List all pairs (x, y) such that $x + y = 2$

Stage 2) List all pairs (x, y) such that $x + y = 3$

…

Stage n) List all pairs (x, y) such that $x + y = n + 1$

…

- Explain why every element of S appears in the list

Ex: Any $(x, y) \in \mathbb{N} \times \mathbb{N}$ will be listed in stage $x + y - 1$

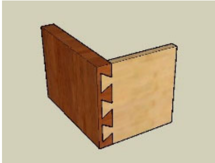
- Define the bijection $f: \mathbb{N} \to S$ by $f(n) =$ the n'th element in this list (ignoring duplicates if needed)

More examples of countable sets

- $\{0,1\}^*$
- $\{\langle M \rangle \mid M \text{ is a Turing machine}\}$
- $\mathbb{Q} = \{\text{rational numbers}\}$

- If $A \subseteq B$ and B is countable, then A is countable
- If A and B are countable, then $A \times B$ is countable

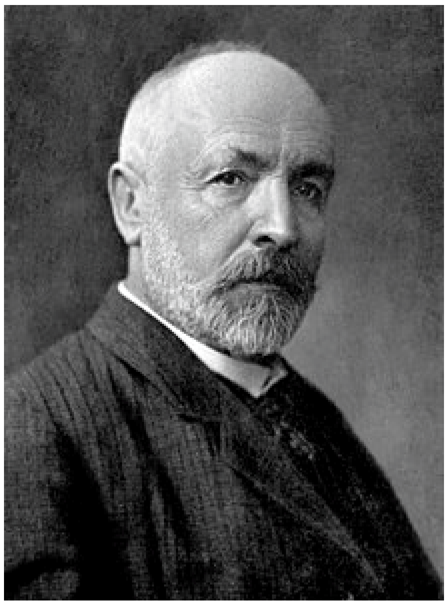
- S is countable if and only if there exists a surjection (an onto function) $f : \mathbb{N} \rightarrow S$

Another version of the dovetailing trick 

Ex: Show that $\mathcal{F} = \{L \subseteq \{0,1\}^* \mid L \text{ is finite}\}$ is countable

So what *isn't* countable?

Cantor’s Diagonalization Method



Georg Cantor 1845-1918

- Invented set theory
- Defined countability, uncountability, cardinal and ordinal numbers, …

Some praise for his work:

“Scientific charlatan…renegade…corruptor of youth”
–L. Kronecker

“Set theory is wrong…utter nonsense…laughable”
–L. Wittgenstein

Uncountability of the reals

Theorem: The real interval $[0,1]$ is uncountable.

Proof: Assume for the sake of contradiction it were countable, and let $f: \mathbb{N} \to [0,1]$ be a bijection

n	$f(n)$
1	$0\,.\,d_1^1\ d_2^1\ d_3^1\ d_4^1\ d_5^1\ \ldots$
2	$0\,.\,d_1^2\ d_2^2\ d_3^2\ d_4^2\ d_5^2\ \ldots$
3	$0\,.\,d_1^3\ d_2^3\ d_3^3\ d_4^3\ d_5^3\ \ldots$
4	$0\,.\,d_1^4\ d_2^4\ d_3^4\ d_4^4\ d_5^4\ \ldots$
5	$0\,.\,d_1^5\ d_2^5\ d_3^5\ d_4^5\ d_5^5\ \ldots$

Construct $b \in [0,1]$ which does not appear in this table – contradiction!

$b = 0.b_1 b_2 b_3 \ldots$ where $b_n \neq d_n^n$ (digit n of $f(n)$)

Uncountability of the reals

A concrete example of the contradiction construction:

n	$f(n)$
1	$0.8675309\ldots$
2	$0.1415926\ldots$
3	$0.7182818\ldots$
4	$0.4444444\ldots$
5	$0.1337133\ldots$

Construct $b \in [0,1]$ which does not appear in this table – contradiction!

$b = 0.b_1 b_2 b_3 \ldots$ where $b_n \neq d_n^n$ (digit n of $f(n)$)

Diagonalization

This process of constructing a counterexample by "contradicting the diagonal" is called diagonalization

Structure of a diagonalization proof

Say you want to show that a set T is uncountable

1) Assume, for the sake of contradiction, that T is countable with bijection $f: \mathbb{N} \to T$
2) "Flip the diagonal" to construct an element $b \in T$ such that $f(n) \neq b$ for every n

Ex: Let $b = 0.b_1 b_2 b_3 \ldots$ where $b_n \neq d_n^n$
(where d_n^n is digit n of $f(n)$)

3) Conclude that f is not onto, which contradicts our assumption that f is a bijection

A general theorem about set sizes

Theorem: Let X be any set. Then the power set $P(X)$ does **not** have the same size as X.

Proof: Assume for the sake of contradiction that there is a bijection $f: X \rightarrow P(X)$

What should we do?

a) Show that for every $S \in P(X)$, there exists $x \in X$ such that $f(x) = S$

b) Construct a set $S \in P(X)$ (meaning, $S \subseteq X$) that cannot be the output $f(x)$ for any $x \in X$

c) Construct a set $S \in P(X)$ and two distinct $x, x' \in X$ such that $f(x) = f(x') = S$

Diagonalization argument

Assume a bijection $f: X \to P(X)$

x					
x_1					
x_2					
x_3					
x_4					
$\vdots$					

Diagonalization argument

Assume a bijection $f: X \to P(X)$

x	$x_1 \in f(x)?$	$x_2 \in f(x)?$	$x_3 \in f(x)?$	$x_4 \in f(x)?$	…
x_1	Y	N	Y	Y	
x_2	N	N	Y	Y	
x_3	Y	Y	Y	N	
x_4	N	N	Y	N	
$\vdots$					$\ddots$

Define S by flipping the diagonal:

Put $x_i \in S \iff x_i \notin f(x_i)$

Example

Let $X = \{1, 2, 3\}$, $P(X) = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}$

Ex. $f(1) = \{1, 2\}$, $f(2) = \emptyset$, $f(3) = \{2\}$

x	$1 \in f(x)$?	$2 \in f(x)$?	$3 \in f(x)$?
1			
2			
3			

Construct $S =$

A general theorem about set sizes

Theorem: Let X be any set. Then the power set $P(X)$ does **not** have the same size as X.

Proof: Assume for the sake of contradiction that there is a bijection $f: X \to P(X)$

Construct a set $S \in P(X)$ that cannot be the output $f(x)$ for any $x \in X$:

$$S = \{x \in X \mid x \notin f(x)\}$$

If $S = f(y)$ for some $y \in X$,

then $y \in S$ if and only if $y \notin S$

Undecidable Languages

Undecidability / Unrecognizability

Definition: A language L is **undecidable** if there is no TM deciding L

Definition: A language L is **unrecognizable** if there is no TM recognizing L

An existential proof

Theorem: There exists an undecidable language over $\{0, 1\}$

Proof:

Set of all encodings of TM deciders: $X \subseteq \{0,1\}^*$

Set of all languages over $\{0, 1\}$:

a) $\{0, 1\}$
b) $\{0, 1\}^*$
c) $P(\{0,1\}^*)$: The set of all subsets of $\{0,1\}^*$
d) $P(P(\{0,1\}^*))$: The set of all subsets of the set of all subsets of $\{0,1\}^*$

An existential proof

Theorem: There exists an undecidable language over $\{0,1\}$

Proof:

Set of all encodings of TM deciders: $X \subseteq \{0,1\}^*$

Set of all languages over $\{0,1\}$: $P(\{0,1\}^*)$

There are more languages than there are TM deciders!

$\Rightarrow$ There must be an undecidable language

An existential proof

Theorem: There exists an unrecognizable language over $\{0,1\}$

Proof:

Set of all encodings of TMs: $X \subseteq \{0,1\}^*$

Set of all languages over $\{0,1\}$: $P(\{0,1\}^*)$

There are more languages than there are TM recognizers!

$\Rightarrow$ There must be an unrecognizable language

“Almost all” languages are undecidable

But how do we actually find one?

An Explicit Undecidable Language

Last time:

Theorem: Let X be any set. Then the power set $P(X)$ does **not** have the same size as X.

1) Assume, for the sake of contradiction, that there is a bijection $f: X \to P(X)$
2) "Flip the diagonal" to construct a set $S \in P(X)$ such that $f(x) \neq S$ for every $x \in X$
3) Conclude that f is not onto, contradicting assumption that f is a bijection

Specializing the proof

Theorem: Let X be the set of all TM deciders. Then there exists an undecidable language in $P(\{0,1\}^*)$

1) Assume, for the sake of contradiction, that $L: X \rightarrow P(\{0,1\}^*)$ is onto
2) "Flip the diagonal" to construct a language $UD \in P(\{0,1\}^*)$ such that $L(M) \neq UD$ for every $M \in X$
3) Conclude that L is not onto, a contradiction

An explicit undecidable language

TM M					
M_1					
M_2					
M_3					
M_4					
$\vdots$					

Why is it possible to enumerate all TMs like this?

a) The set of all TMs is finite
b) The set of all TMs is countably infinite
c) The set of all TMs is uncountable

An explicit undecidable language

TM M	$M(\langle M_1 \rangle)$?	$M(\langle M_2 \rangle)$?	$M(\langle M_3 \rangle)$?	$M(\langle M_4 \rangle)$?		$D(\langle D \rangle)$?
M_1	Y	N	Y	Y	…	
M_2	N	N	Y	Y		
M_3	Y	Y	Y	N		
M_4	N	N	Y	N		
$\vdots$					$\ddots$	
D						

$UD = \{\langle M \rangle \mid M$ is a TM that does not accept on input $\langle M \rangle\}$

Claim: UD is undecidable

An explicit undecidable language

Theorem: $UD = \{\langle M \rangle \mid M \text{ is a TM that does } \textbf{not} \text{ accept on input } \langle M \rangle\}$ is undecidable

Proof: Suppose for contradiction, that TM D decides UD