BU CS 332 – Theory of Computation

Lecture 17:

- Test 2 Review

Mark Bun

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Test 2 Topics

Turing Machines (3.1, 3.3)

- Know the three different “levels of abstraction” for defining Turing machines and how to convert between them: Formal/state diagram, implementation-level, and high-level
- Know the definition of a configuration of a TM and the formal definition of how a TM computes
- Know how to “program” Turing machines by giving state diagrams and implementation-level descriptions
- Understand the Church-Turing Thesis

TM Variants (3.2)

- Understand the following TM variants: TM with stay-put, TM with two-way infinite tape, Multi-tape TMs, Nondeterministic TMs
- Know how to give a simulation argument (implementation-level and high-level description) to compare the power of TM variants
- Understand the specific simulation arguments we've seen: two-way infinite TM by basic TM, multi-tape TM by basic TM, nondeterministic TM by basic TM

Decidability (4.1)

- Understand how to use a TM to simulate another machine (DFA, another TM)
- Know the specific decidable languages from language theory that we've discussed, and how to decide them: $A_{DFA}, E_{DFA}, EQ_{DFA}$, etc.
- Know how to use a reduction to one of these languages to show that a new language is decidable

Undecidability (4.2)

- Know the definitions of countable and uncountable sets and how to prove countability and uncountability
- Understand how diagonalization is used to prove the existence of an explicit undecidable language UD
- Know that a language is decidable iff it is recognizable and its complement is recognizable, and understand the proof

A_{TM} is undecidable + A_{TM} recognizable

$\Rightarrow$ $\overline{A_{TM}}$ is unrecognizable

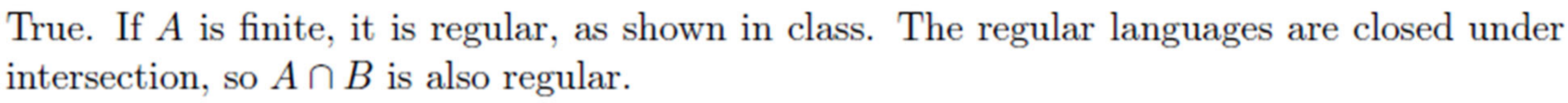
Reducibility (5.1)

- Understand how to use a reduction (contradiction argument) to prove that a language is undecidable
- Know the reductions showing that $HALT_{TM}$, $E_{TM}, REGULAR_{TM}, EQ_{TM}$ are undecidable

- You are not responsible for understanding the computation history method.
- I will not ask you to prove that something is undecidable by reduction on our in-class test. But you may get a conceptual question about how reductions work / what they can be used to show.

True or False

- It’s all about the justification!
- The logic of the argument has to be clear
- Restating the question is not justification; we’re looking for additional insight
- False statements should be justified by a specific counterexample whenever possible.

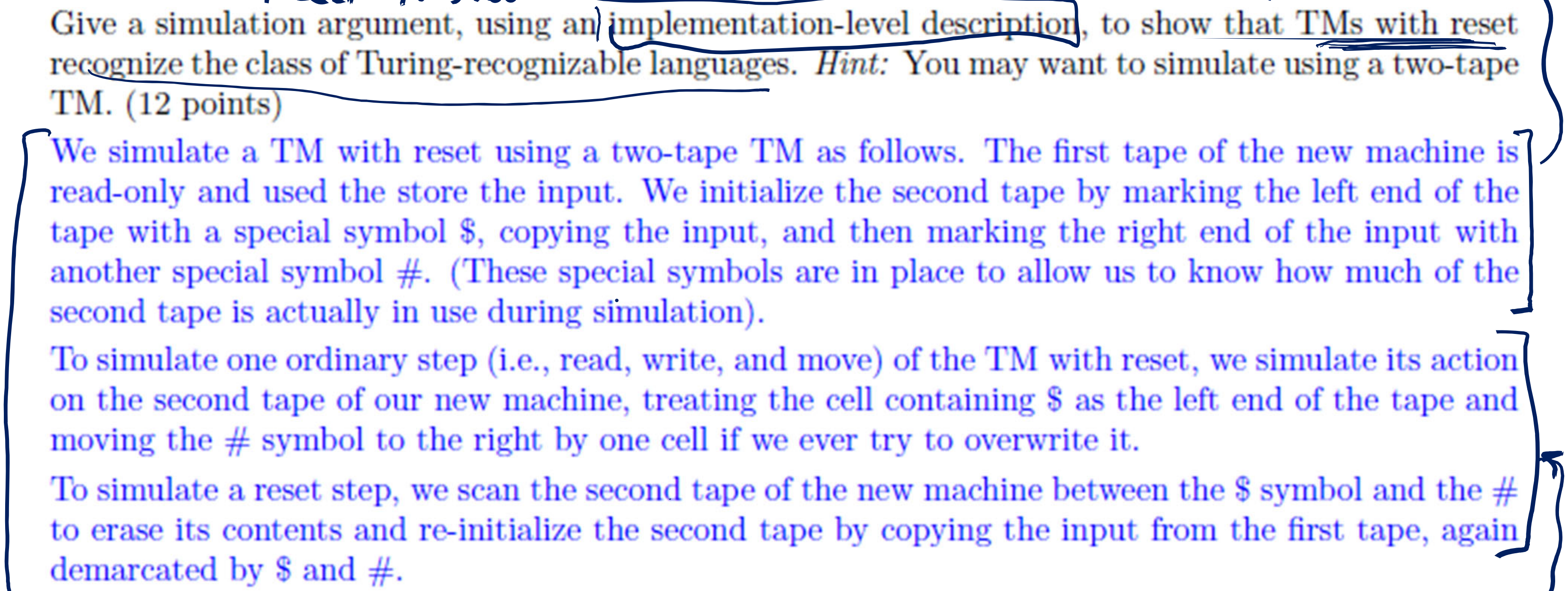


True. If A is finite, it is regular, as shown in class. The regular languages are closed under intersection, so $A \cap B$ is also regular.

Simulation arguments, constructing deciders

initialize input

other direction: Show TM w/ reset is at least as powerful as basic TM
Need to show how to simulate basic TM on TM w/ reset



Give a simulation argument, using an implementation-level description, to show that TMs with reset recognize the class of Turing-recognizable languages. *Hint:* You may want to simulate using a two-tape TM. (12 points)

We simulate a TM with reset using a two-tape TM as follows. The first tape of the new machine is read-only and used the store the input. We initialize the second tape by marking the left end of the tape with a special symbol \$, copying the input, and then marking the right end of the input with another special symbol #. (These special symbols are in place to allow us to know how much of the second tape is actually in use during simulation).

To simulate one ordinary step (i.e., read, write, and move) of the TM with reset, we simulate its action on the second tape of our new machine, treating the cell containing \$ as the left end of the tape and moving the # symbol to the right by one cell if we ever try to overwrite it.

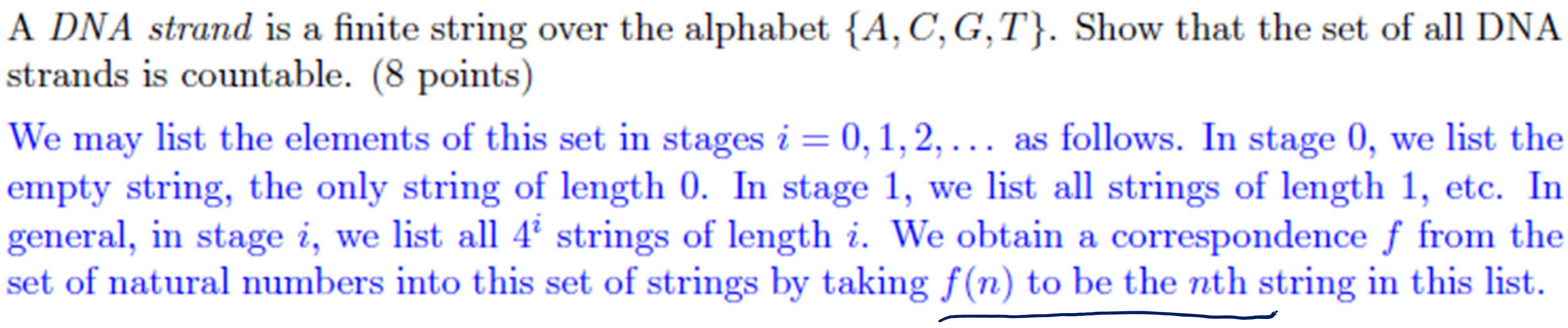
To simulate a reset step, we scan the second tape of the new machine between the \$ symbol and the # to erase its contents and re-initialize the second tape by copying the input from the first tape, again demarcated by \$ and #.

Simulate a TM w/ reset using a basic TM
→ TM w/ reset is at most as powerful as basic TM

Explain how to simulate one move

- Full credit for a clear and correct description of the new machine
- Can still be a good idea to provide an explanation (partial credit, clarifying ambiguity)

Countability proofs

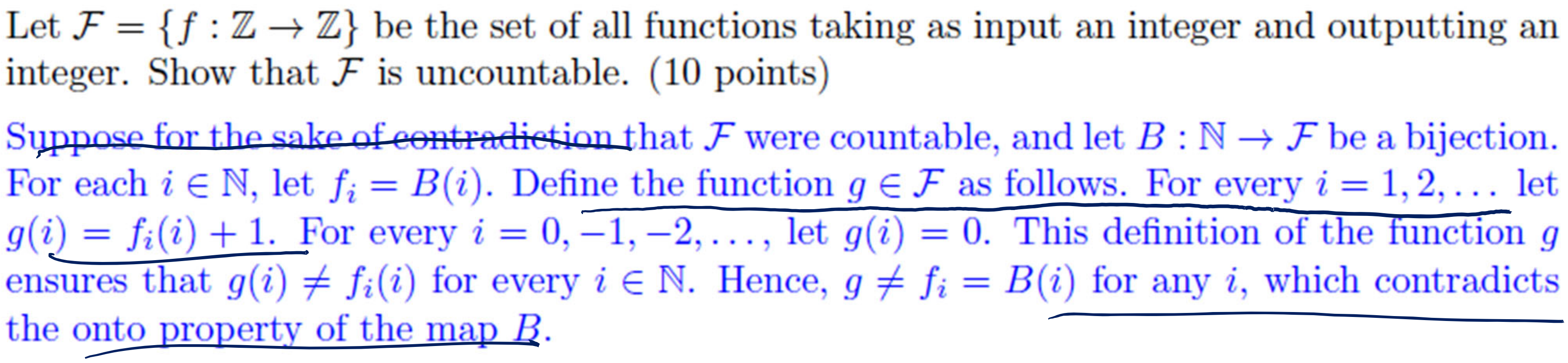


A *DNA strand* is a finite string over the alphabet $\{A, C, G, T\}$. Show that the set of all DNA strands is countable. (8 points)

We may list the elements of this set in stages $i = 0, 1, 2, \ldots$ as follows. In stage 0, we list the empty string, the only string of length 0. In stage 1, we list all strings of length 1, etc. In general, in stage i, we list all 4^i strings of length i. We obtain a correspondence f from the set of natural numbers into this set of strings by taking $f(n)$ to be the nth string in this list.

- Describe how to list all the elements in your set, usually in a succession of finite "stages"
- Describe how this listing process gives you a bijection from the natural numbers

Uncountability proofs

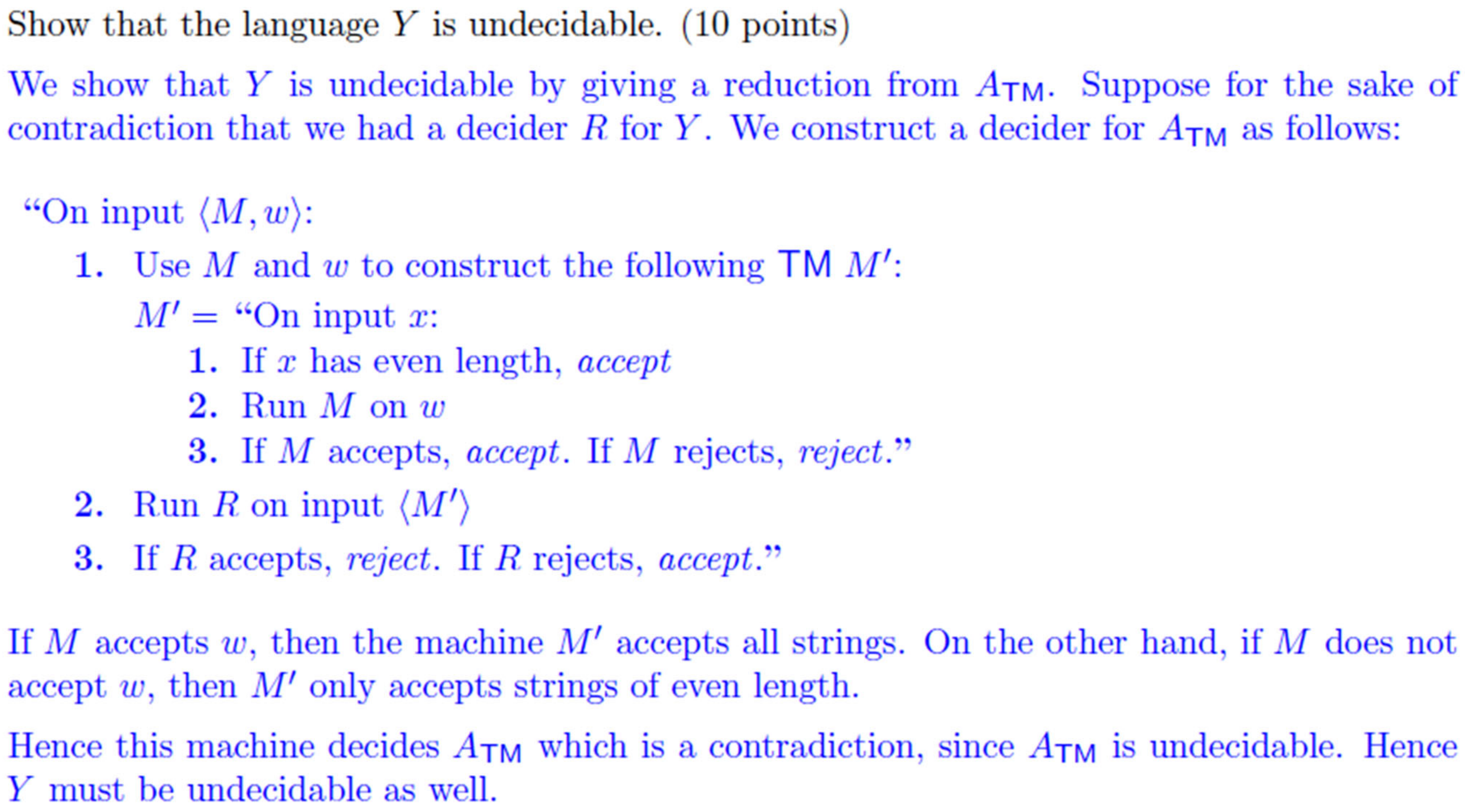


Let $\mathcal{F} = \{f : \mathbb{Z} \to \mathbb{Z}\}$ be the set of all functions taking as input an integer and outputting an integer. Show that $\mathcal{F}$ is uncountable. (10 points)

Suppose for the sake of contradiction that $\mathcal{F}$ were countable, and let $B : \mathbb{N} \to \mathcal{F}$ be a bijection. For each $i \in \mathbb{N}$, let $f_i = B(i)$. Define the function $g \in \mathcal{F}$ as follows. For every $i = 1, 2, \dots$ let $g(i) = f_i(i) + 1$. For every $i = 0, -1, -2, \dots,$ let $g(i) = 0$. This definition of the function g ensures that $g(i) \neq f_i(i)$ for every $i \in \mathbb{N}$. Hence, $g \neq f_i = B(i)$ for any i, which contradicts the onto property of the map B.

- The 2-D table is useful for helping you think about diagonalization, but does not need to appear in the proof
- The essential part of the proof is the construction of the "inverted diagonal" element, and the proof that it works

Undecidability proofs



Show that the language Y is undecidable. (10 points)

We show that Y is undecidable by giving a reduction from A_{TM}. Suppose for the sake of contradiction that we had a decider R for Y. We construct a decider for A_{TM} as follows:

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"On input $\langle M, w\rangle$:
   1. Use $M$ and $w$ to construct the following TM $M'$:
      $M'$ = "On input $x$:
         1. If $x$ has even length, accept
         2. Run $M$ on $w$
         3. If $M$ accepts, accept. If $M$ rejects, reject."
   2. Run $R$ on input $\langle M'\rangle$
   3. If $R$ accepts, reject. If $R$ rejects, accept."
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If M accepts w, then the machine M' accepts all strings. On the other hand, if M does not accept w, then M' only accepts strings of even length.

Hence this machine decides A_{TM} which is a contradiction, since A_{TM} is undecidable. Hence Y must be undecidable as well.

Practice Problems

Decidability and Recognizability

Let $E_{DFA} = \{\langle D\rangle \mid D$ is a DFA and $L(D) = \emptyset\}$

$A = \{\langle R, S\rangle \mid R, S$ are regexes such that $L(R) \subseteq L(S)\}$

Show that A is decidable Construct a decider for A:

TM M: On input $\langle R, S\rangle$ where R, S are regexes

1) Convert R and S to DFAs D_R and D_S respectively

2) Use regularity closure constructions to build DFA D s.t. $L(D) = L(R) \setminus L(S) = L(R) \cap \overline{L(S)}$

3) Run decider for E_{DFA} on $\langle D\rangle$. If it accepts, accept. If rejects, reject.

Correctness:

If $\langle R, S\rangle \in A$, then $L(R) \subseteq L(S) \Rightarrow L(D) = L(R) \setminus L(S) = \emptyset$

$\Rightarrow$ Decider for E_{DFA} accepts $\Rightarrow$ M accepts ✓

If $\langle R, S\rangle \notin A$, then $L(R) \not\subseteq L(S) \Rightarrow \exists\ w \in L(R)$, but $w \in L(S)$

$\Rightarrow L(D) = L(R) \setminus L(S) \neq \emptyset \Rightarrow$ Decider for E_{DFA} rejects $\Rightarrow$ M rejects ✓

Another proof of decidability:

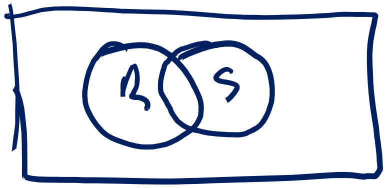
$EQ_{DFA} = \{\langle D_1, D_2\rangle \mid$ DFAs D_1, D_2 $L(D_1) = L(D_2)\}$

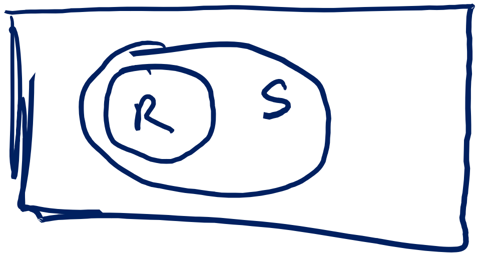
On input $\langle R, S\rangle$:

1) Convert R, S to DFAs D_R, D_S
2) Construct DFA D s.t. $L(D) = L(R) \cap L(S)$
3) Run decider for EQ_{DFA} on input $\langle D, D_R\rangle$. If accepts, accept. If rejects, reject.

If $L(R) \subseteq L(S) \Rightarrow L(D) = L(R) \cap L(S) = L(R)$

If $L(R) \not\subseteq L(S) \Rightarrow L(R) \cap L(S) \neq L(R)$





Assume alphabet is $\{0,1\}$

Prove that $\overline{E_{\text{TM}}}$ is recognizable

$E_{TM} = \{ \langle M \rangle \mid M$ is a TM and $L(M) = \emptyset \}$

$\overline{E_{TM}} = \{ \langle M \rangle \mid M$ is a TM and $L(M) \neq \emptyset \}$

$= \{ \langle M \rangle \mid M$ is a TM, and $\exists\, w$ s.t. M accepts $w \}$

Design an NTM N recognizing $\overline{E_{TM}}$:

On input $\langle M \rangle$:

1) Nondeterministically guess $w \in \{0,1\}^*$.

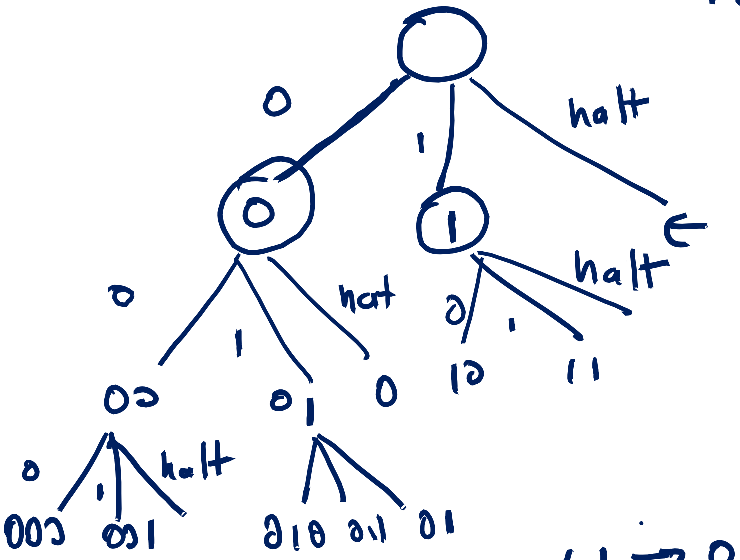
2) Run M on input w. If accepts, accept, if rejects, reject.

Correctness:

If $\langle M \rangle \in \overline{E_{TM}}$, $\exists\, w$ s.t. $M(w)$ accepts $\Rightarrow$ branch where w was guessed leads N to accept ✓

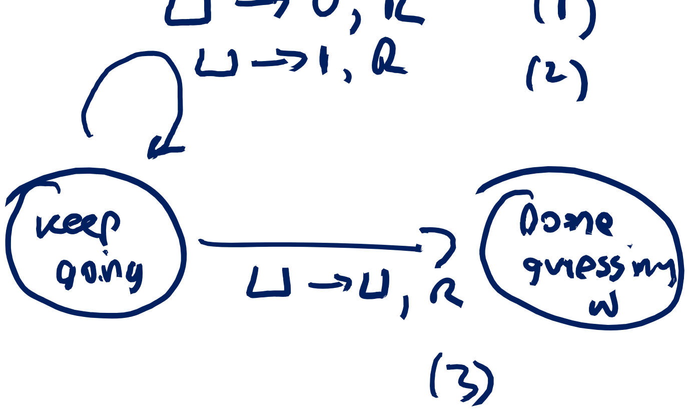
If $\langle M \rangle \notin \overline{E_{TM}}$, then all w lead $M(w)$ to not accept $\Rightarrow N$ does not accept.

Not a decider

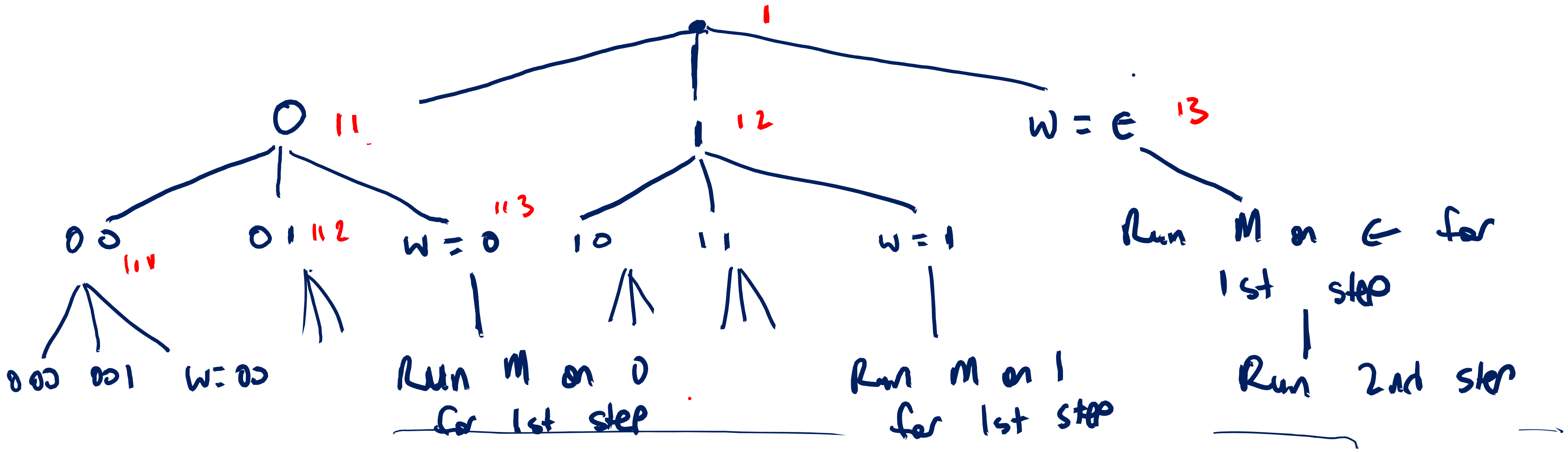


How to nondeterministically guess $w \in \{0,1\}^*$?

Some branches may loop forever



Simulating NTM using a (multi-tape) TM



Explicit unrolling w/ dovetailing trick:

For each string $w_1, w_2, w_3, \ldots \in \{0,1\}^*$:

Run M on on $w_1, \ldots, w_i$ for i steps each.

If any accept, accept

Prove that if A and B are decidable, then so is $A \setminus B$

Countable and Uncountable Sets

Show that the set of all valid (i.e., compiling without errors) C++ programs is countable

A Celebrity Twitter Feed is an infinite sequence of ASCII strings, each with at most 140 characters. Show that the set of Celebrity Twitter Feeds is uncountable.

Undecidability and Unrecognizability

Prove or disprove: If A and B are recognizable, then so is $A \setminus B$

Prove that the language $ALL_{\mathrm{TM}} = \{\langle M\rangle | M \text{ is a TM and } L(M) = \Sigma^*\}$ is undecidable