

# BU CS 332 – Theory of Computation

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## Lecture 20:

- Time/Space Hierarchies
- Complexity Class P

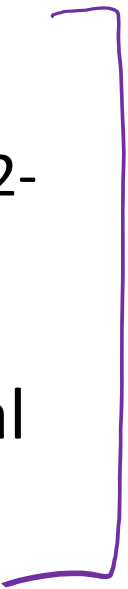
Reading:

Sipser Ch 9.1, 7.2

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# Last Time

- Asymptotic notation
  - Analyzing time / space usage of Turing machines (algorithms)
  - Multi-tape TMs can solve problems faster than single-tape TMs  
E.g.,  $A = \{0^m 1^m \mid m \geq 0\}$  can be decided in  $O(n)$  time on a 2-tape TM, but cannot be decided in  $o(n \log n)$  time on a basic TM
  - Complexity class P: Languages decidable in polynomial time
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# Time complexity

**Time complexity** of a TM (algorithm) = maximum number of steps it takes on a worst-case input

Formally: Let  $f : \mathbb{N} \rightarrow \mathbb{N}$ . A TM  $M$  runs in time  $f(n)$  if on every input  $w \in \Sigma^n$ ,  $M$  halts on  $w$  within at most  $f(n)$  steps

Set (class) of languages

A language  $A \in \text{TIME}(f(n))$  if there exists a basic single-tape (deterministic) TM  $M$  that

- 1) Decides  $A$ , and
- 2) Runs in time  $O(f(n))$

# Single vs. Multi-Tape

**Theorem:** Let  $t(n) \geq n$  be a function. Every multi-tape TM running in time  $t(n)$  has an equivalent single-tape TM running in time  $O(t(n)^2)$

Suppose  $B$  is decidable in time  $O(n^2)$  on a 42-tape TM. What is the best upper bound you can give on the runtime of a basic single-tape TM deciding  $B$ ?

- a)  $O(n^2)$
- b)  $O(n^4)$
- c)  $O(n^{84})$
- d)  $2^{O(n)}$

$$O(t(n)^2) = O((n^2)^2) = O(n^4)$$



# Single vs. Multi-Tape

**Theorem:** Let  $t(n) \geq n$  be a function. Every multi-tape TM running in time  $t(n)$  has an equivalent single-tape TM running in time  $O(t(n)^2)$

**Proof idea:**

We already saw how to **simulate** a multi-tape TM with a single-tape TM

Need a runtime analysis of this construction

# Simulating Multiple Tapes

(Implementation-Level Description)

On input  $w = w_1 w_2 \dots w_n$  *5 tape cells used*

1. Format tape into  $\# \overbrace{w_1 w_2 \dots w_n} \# \sqcup \# \sqcup \# \dots \#$

2. For each move of  $M$ : *Original multi-tape TM*

Scan left-to-right, finding current symbols  *$O(s)$  time*

Scan left-to-right, writing new symbols,  *$O(s)$  time*

Scan left-to-right, moving each tape head  *$O(s)$  time*

If a tape head goes off the right end, insert blank

If a tape head goes off left end, move back right

# Single vs. Multi-Tape

**Theorem:** Let  $t(n) \geq n$  be a function. Every  $\overbrace{\text{multi-tape}}^{k\text{-tape}}$  TM running in time  $t(n)$  has an equivalent single-tape TM running in time  $O(t(n)^2)$

**Proof:** Time analysis of simulation

- Time to initialize (i.e., format tape):  $O(n + k)$
- Time to simulate one step of multi-tape TM:  $O(k \cdot t(n))$

Multi-tape TM uses  $\leq k \cdot t(n)$  tape cells

3 linear scans of single-tape TM takes  $O(k \cdot t(n))$  steps

- Number of  $\overbrace{\text{steps}}^{\text{multi-tape}}$  to simulate:  $\underline{t(n)}$

$\Rightarrow$  Total time:  $O(n+k) + \underline{t(n)} \cdot O(k \cdot t(n)) = O(t(n)^2)$   
when  $k$  is constant

# Extended Church-Turing Thesis

Every “reasonable” (physically realizable) model of computation can be simulated by a basic, single-tape TM with only a **polynomial** slowdown.

E.g., doubly infinite TMs, multi-tape TMs, RAM TMs

Does **not** include nondeterministic TMs (not reasonable)

**Possible counterexamples?** Randomized computation, parallel computation, DNA computing, quantum computation

More believable: All deterministic, sequential models can be simulated by basic TM w/ polynomial slowdown.

# Space complexity

**Space complexity** of a TM (algorithm) = maximum number of tape cells it uses on a worst-case input

Formally: Let  $f : \mathbb{N} \rightarrow \mathbb{N}$ . A TM  $M$  runs in space  $f(n)$  if on **every** input  $w \in \Sigma^n$ ,  $M$  halts on  $w$  using at most  $f(n)$  cells

set of languages "complexity class"

A language  $A \in \text{SPACE}(f(n))$  if there exists a basic single-tape (deterministic) TM  $M$  that

- 1) Decides  $A$ , and
- 2) Runs in space  $O(f(n))$

# How does space relate to time?



Which of the following is true for every function

$f(n) \geq n$ ?  $TIME(f(n)) =$  set of all languages decidable in  $O(f(n))$  time  
 $SPACE(f(n)) =$  " " " " space

- a)  $TIME(f(n)) \subseteq SPACE(f(n))$  let  $A \in TIME(f(n))$
- b)  $SPACE(f(n)) \subseteq TIME(f(n))$  Then  $\exists$  TM  $M$  for  $A$
- c)  $TIME(f(n)) = SPACE(f(n)) \Rightarrow M$  runs in  $O(f(n))$  time
- d) None of the above  $\Rightarrow A \in SPACE(f(n))$

$\{Hopcroft - Paul Valiant '77\}$

$$TIME(f(n)) \subseteq SPACE(f(n) / \log f(n))$$

E.g.  $TIME(n^2) \subseteq SPACE(n^2 / \log n), \text{ i.e. } \text{If } A \text{ can be decided in time } n^2, \text{ can be decided } n^2 / \log n$

## Back to our example

$$A = \{0^m 1^m \mid m \geq 0\} \quad A \in \text{TIME}(n \log n)$$

$M$  = "On input  $w$ :

$$A \in \text{SPACE}(n)$$

1. Scan input and reject if not of the form  $0^*1^*$
2. While input contains both 0's and 1's:  
Cross off one 0 and one 1
3. **Accept** if no 0's and no 1's left. Otherwise, **reject**."

**Theorem:** Let  $s(n) \geq n$  be a function. Every multi-tape TM running in space  $s(n)$  has an equivalent single-tape TM running in space  $O(s(n))$

# Hierarchy Theorems

# More time, more problems

We know, e.g., that  $TIME(n^2) \subseteq TIME(n^3)$

(Anything we can do in quadratic time we can do in cubic time)

**Question:** Are there problems that we can solve in cubic time that we cannot solve in quadratic time?

**Theorem:** There is a language  $L \in TIME(n^3)$ ,  
but  $L \notin TIME(n^2)$

“Time hierarchy”:

$$TIME(n) \subsetneq TIME(n^2) \subsetneq TIME(n^3) \subsetneq TIME(n^4) \dots$$

“is a subset of, but is not equal to”

$A \subsetneq B$  means  $A \subset B$  and  $\exists x \in B$  but  $x \notin A$ .

# Diagonalization redux

TM $M$	$M(\langle M_1 \rangle)$ ?	$M(\langle M_2 \rangle)$ ?	$M(\langle M_3 \rangle)$ ?	$M(\langle M_4 \rangle)$ ?		$D(\langle D \rangle)$ ?
$M_1$	<del>Y</del> $N$	N	Y	Y	...	
$M_2$	N	<del>N</del> $Y$	Y	Y		
$M_3$	Y	Y	<del>Y</del> $N$	N		
$M_4$	N	N	Y	N		
$\vdots$					$\ddots$	
$D$						

$UD = \{\langle M \rangle \mid M \text{ is a TM that does not accept input } \langle M \rangle\}$

$L = \{\langle M \rangle \mid M \text{ is a TM that does not accept input } \langle M \rangle$   
**within  $n^{2.5}$  steps**  $n = \langle M \rangle$

# An explicit separating language

**Theorem:**  $L = \{\langle M \rangle \mid M \text{ is a TM that does not accept input } \langle M \rangle \text{ within } n^{2.5} \text{ steps}\}$

is in  $TIME(n^3)$ , but not in  $TIME(n^2)$

**Proof Sketch:** In  $TIME(n^3)$

On input  $\langle M \rangle$ :

- takes  $O(n^{2.5} \log n)$  time  $\leq O(n^3)$
1. Simulate  $M$  on input  $\langle M \rangle$  for  $n^{2.5}$  steps
  2. If  $M$  accepts, **reject**. If  $M$  rejects or did not yet halt, **accept**.

$\langle M \rangle \in L \Rightarrow M(\langle M \rangle)$  does not accept w/in  $n^{2.5}$  steps  $\Rightarrow$  alg. **accepts**

$\langle M \rangle \notin L \Rightarrow M(\langle M \rangle)$  accepts w/in  $n^{2.5}$  steps  $\Rightarrow$  alg. **rejects**.

# An explicit separating language

**Theorem:**  $L = \{\langle M \rangle \mid M \text{ is a TM that does not accept input } \langle M \rangle \text{ within } n^{2.5} \text{ steps}\}$

is in  $TIME(n^3)$ , but not in  $TIME(n^2)$

**Proof Sketch:** Not in  $TIME(n^2)$

Suppose for contradiction that  $D$  decides  $L$  in time  $O(n^2)$

Case 1: If  $\langle D \rangle \in L \Rightarrow D$  does not accept  $\langle D \rangle$  in  $n^{2.5}$  steps  
 $\Rightarrow$  contradicts assumption  $D$  accepts  $\langle D \rangle$  in time  $O(n^2)$  ✗

Case 2: If  $\langle D \rangle \notin L \Rightarrow D$  accepts  $\langle D \rangle$  w/in  $n^{2.5}$  steps  
contradicts assumption  $D$  rejects  $\langle D \rangle$  ✗

# Time and space hierarchy theorems

- For every\* function  $t(n) \geq n \log n$ , a language exists that is decidable in  $t(n)$  time, but not in  $o\left(\frac{t(n)}{\log t(n)}\right)$  time.

Ex:  $\text{TIME}(n^3) \not\subseteq \text{TIME}\left(\frac{n^3}{\log^2 n}\right)$  because  $\frac{n^3}{\log^2 n} = o\left(\frac{n^3}{\log(n^3)}\right)$

- For every\* function  $s(n) \geq \log n$ , a language exists that is decidable in  $s(n)$  space, but not in  $o(s(n))$  space.

\*“time constructible” and “space constructible”, respectively

# Complexity Class P

# Time and space complexity

## The basic questions

1. How do we measure complexity?
2. Asymptotic notation
3. How robust is the TM model when we care about measuring complexity?
4. How do we mathematically capture our intuitive notion of “efficient algorithms”?

# Complexity class P

**Definition:** P is the class of languages decidable in polynomial time on a basic single-tape (deterministic) TM

$$P = \bigcup_{k=1}^{\infty} \text{TIME}(n^k) = \text{TIME}(n) \cup \text{TIME}(n^2) \cup \text{TIME}(n^3) \cup \dots$$

- Class doesn't change if we substitute in another reasonable deterministic model (Extended Church-Turing)
- **Cobham-Edmonds Thesis:** Roughly captures class of problems that are feasible to solve on computers

# Check your type checker: P



$P$ : "functions polynomial time" =  $\{ f: \Sigma^* \rightarrow \Sigma^* \mid f \text{ computable in poly time} \}$

Consider the following computational problem: Given two numbers  $x, y$  (written in binary), output their sum

$x + y$  (in binary). Which of the following is true?

$$S = \{ \langle x, y, z \rangle \mid x + y = z \} \in P$$

- ☒ a) This is a problem in P  $\{ \langle x, i \rangle \mid i \text{th bit of } f(x) = 1 \}$
- ☒ b) This problem is not in P because it cannot be solved by a Turing machine (i.e., it's undecidable)
- ☒ c) This problem is not in P because it cannot be solved in polynomial time
- d) This problem is not in P because it is not a decision problem (i.e., does not correspond to a language)

## A note about encodings

We'll still use the notation  $\langle \cdot \rangle$  for “any reasonable” encoding of the input to a TM...but now we have to be more careful about what we mean by “reasonable”

How long is the encoding of a  $V$ -vertex,  $E$ -edge graph...

... as an adjacency matrix?  $O(|V|^2)$

... as an adjacency list?  $O(|V| + |E|)$

How long is the encoding of a natural number  $k$

... in binary?  $|\langle k \rangle| = \log_2 k + 1$

... in decimal?  $|\langle k \rangle| = \log_{10} k + 1 = O(\log_2 k)$

... in unary?  $\langle k \rangle = \underbrace{111\dots1}_k$   $|\langle k \rangle| = \underline{\underline{k}}$

# Describing and analyzing polynomial-time algorithms

- Due to Extended Church-Turing Thesis, we can still use high-level descriptions on multi-tape machines
- Polynomial-time is **robust under composition**:  $\text{poly}(n)$  executions of  $\text{poly}(n)$ -time subroutines run on  $\text{poly}(n)$ -size inputs gives an algorithm running in  $\text{poly}(n)$  time.  
⇒ Can freely use algorithms we've seen before as subroutines if we've analyzed their runtime
- Need to be careful about size of inputs! (Assume inputs represented in binary unless otherwise stated.)