BU CS 332 – Theory of Computation

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Lecture 22:

- P Examples
- NP

Reading:

Sipser Ch 7.2-7.3

HW 9 due Th. Dec. 8

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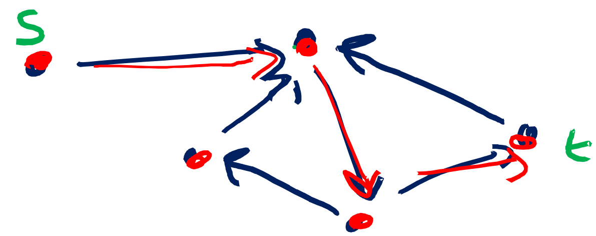
Complexity class P

Definition: P is the class of languages decidable in polynomial time on a basic single-tape (deterministic) TM

$$\mathrm{P} = \bigcup_{k=1}^{\infty} \mathrm{TIME}(n^k)$$

- Class doesn't change if we substitute in another reasonable deterministic model (Extended Church-Turing)
- Cobham-Edmonds Thesis: Roughly captures class of problems that are feasible to solve on computers

Examples of languages in $\mathbf{P}$



$PATH =$
$\{\langle G, s, t\rangle \mid G$ is a directed graph with a directed path from s to $t\}$

Idea: Breadth-first search

Assume G presented as adjacency matrix

Input length:

"On input $\langle G, s, t\rangle$: $\underbrace{|V|^2}_{G} + \underbrace{\log|V|}_{s} + \underbrace{\log|V|}_{t}$

1. Mark start vertex s $= |V|^2 + 2\log|V| =: n$
2. For $i = 1, 2, \ldots, |V|$: runs ~~for~~ $|V|$ steps
3. Mark all neighbors of currently marked vertices $O(|V|^2)$
4. If t is marked, accept. Else, reject."

Total run time:
$|V| \cdot O(|V|^2) = O(|V|^3)$
$= O(n^{3/2})$

Correctness: If $\exists$ an s-t path in G, $\exists$ such a path using $\leq |V|$ vertices. Then t gets marked after $|V|$ steps of BFS $\Rightarrow$ alg. accepts.
If no s-t path, then t will never get marked, so alg. rejects.

Examples of languages in P

$E_{\text{DFA}} = \{\langle D \rangle \mid D \text{ is a DFA that recognizes the empty language}\}$

$E_{DFA} \in P$

Because BFS solves problem in poly-time

Can also reduce to PATH:

For all accept states t of D:

Check if $\langle D, s, t \rangle \in PATH$. If any are, reject

If all are not, accept.

DFA start state

Examples of languages in P

- $RELPRIME = \{\langle x, y\rangle \mid x \text{ and } y \text{ are relatively prime}\}$

  i.e., is $\gcd(x,y) = 1$?

  Euclidean algorithm

- $PRIMES = \{\langle x\rangle \mid x \text{ is prime}\}$

2006 Gödel Prize citation



The 2006 Gödel Prize for outstanding articles in theoretical computer science is awarded to Manindra Agrawal, Neeraj Kayal, and Nitin Saxena for their paper "PRIMES is in P."

In August 2002 one of the most ancient computational problems was finally solved….

A polynomial-time algorithm for $PRIMES$?

Consider the following algorithm for $PRIMES$

On input $\langle x \rangle$: x is written in binary $n = |\langle x \rangle|$

$n = \lceil \log_2 x \rceil$

For $b = 2, 3, 4, 5, \ldots, \sqrt{x}$: $\iff x \approx 2^n$

- Try to divide x by b

- If b divides x, reject # divisions $\approx \sqrt{2^n} = (2^n)^{1/2}$

If all b fail to divide x, accept $= 2^{n/2}$

How many divisions does this algorithm require in terms of $n = |\langle x \rangle|$? a) $O(\sqrt{n})$ b) $O(n)$ c) $2^{O(\sqrt{n})}$ d) $2^{O(n)}$

Beyond polynomial time

Definition: EXP is the class of languages decidable in exponential time on a basic single-tape (deterministic) TM

$$\mathrm{EXP} = \bigcup_{k=1}^{\infty} \mathrm{TIME}(2^{n^k})$$

$$= \mathrm{TIME}(2^{n}) \cup \mathrm{TIME}(2^{n^2}) \cup \mathrm{TIME}(2^{n^3}) \cup \ldots$$

Why study P ?

Criticism of the Cobham-Edmonds Thesis:

- Algorithms running in time n^{100} aren't really efficient

  Response: Runtimes improve with more research

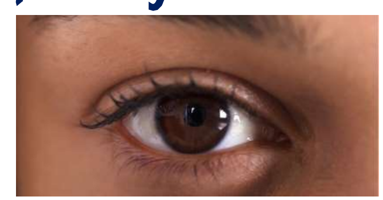
- Does not capture some physically realizable models using randomness, quantum mechanics

  Response: Randomness may not change P, useful principles

"Hardness vs. randomness": If "hard" functions exist, then all randomized algs. can be "derandomized" w/ poly slowdown

$TIME(n)$ vs. $TIME(n^2)$

P vs. EXP



decidable vs. undecidable

Nondeterministic Time and NP

Extended Church-Turing Thesis

Every “reasonable” (physically realizable) model of computation can be simulated by a basic, single-tape TM with only a **polynomial** slowdown.

E.g., doubly infinite TMs, multi-tape TMs, RAM TMs

Does not include nondeterministic TMs (not reasonable)

Nondeterministic time

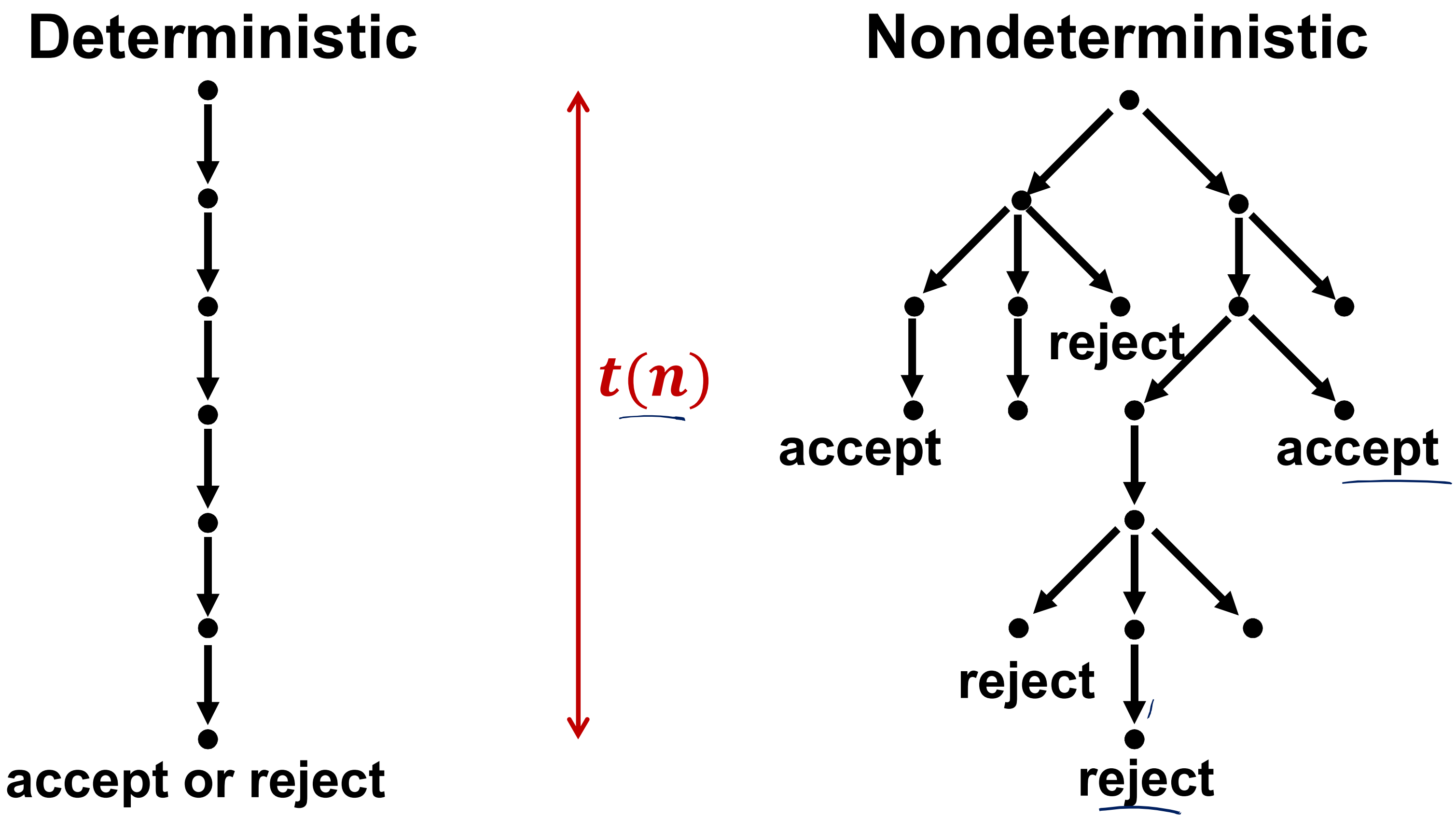
input length runtime

Let $t\colon \underset{\text{input length}}{\mathbb{N}} \to \underset{\text{runtime}}{\mathbb{N}}$

A NTM M runs in time $t(n)$ if on every input $w \in \Sigma^n$,

M halts on w within at most $\underline{t(n)}$ steps on every computational branch

Deterministic vs. nondeterministic time

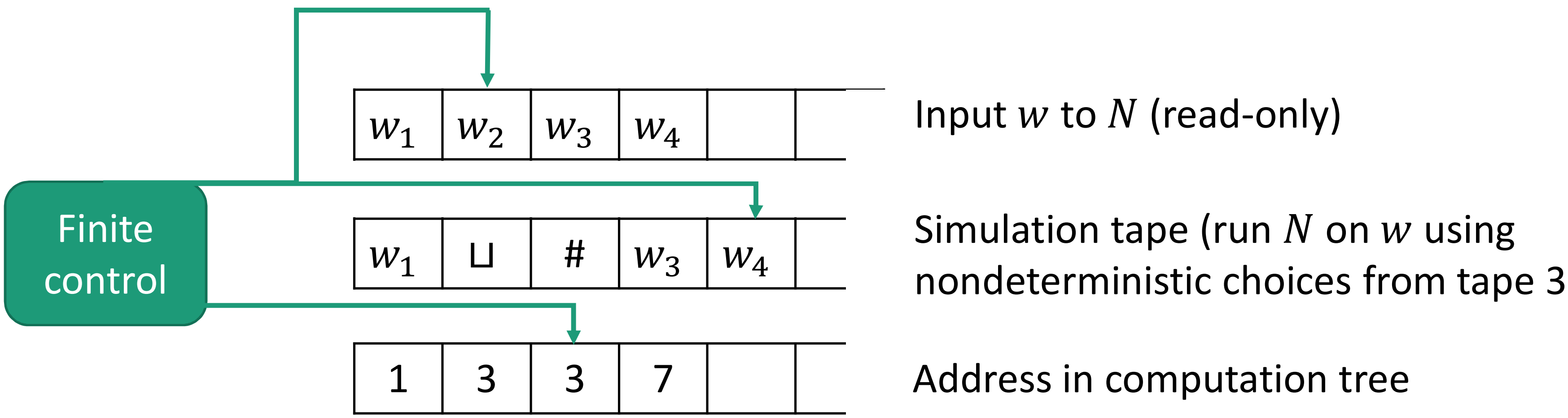


Deterministic vs. nondeterministic time

Theorem: Let $t(n) \geq n$ be a function. Every NTM running in time $t(n)$ has an equivalent single-tape TM running in time $2^{O(t(n))}$

deterministic

Proof: Simulate NTM by 3-tape TM



Counting leaves

What is an upper bound on the maximum number of nodes in a tree with branching factor b and depth t?

a) bt

b) b^t

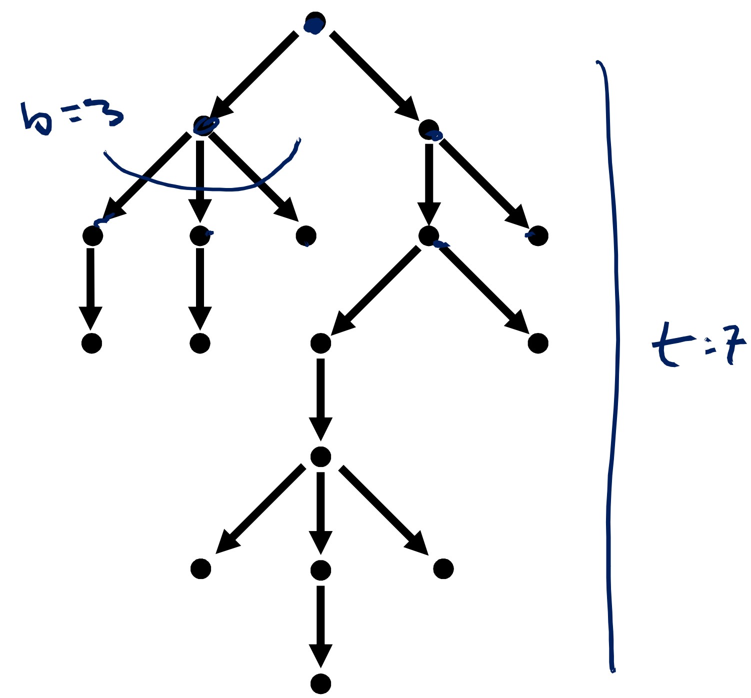
c) t^b

d) 2^t

$$1 + b + b^2 + \dots + b^{t-1}$$

root, depth 2, depth 3, depth t

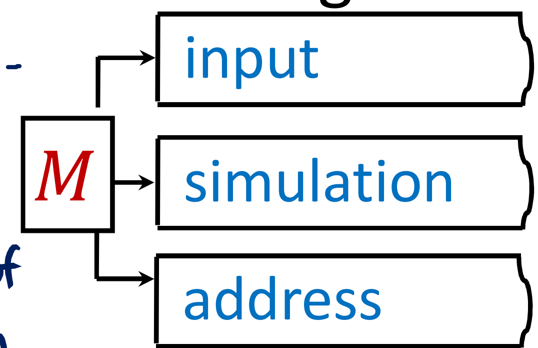
$$= \frac{b^t - 1}{b-1} \le b^t$$



$b^a = (2^{\log b})^a = 2^{a \log b}$

Deterministic vs. nondeterministic time

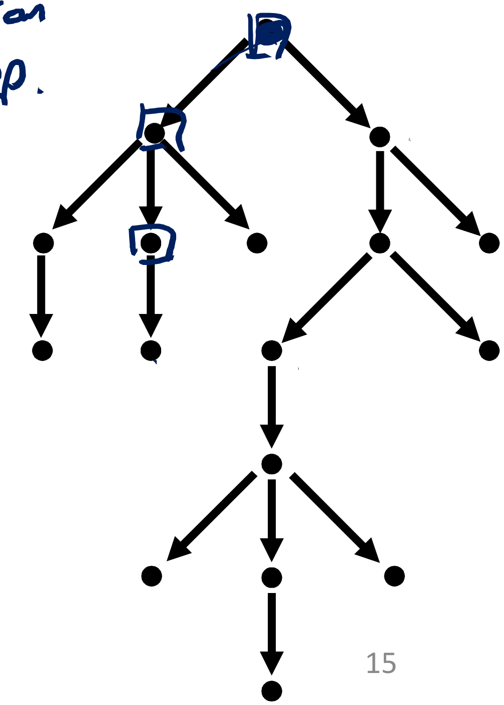
Theorem: Let $t(n) \geq n$ be a function. Every NTM running in time $t(n)$ has an equivalent single-tape TM running in time $2^{O(t(n))}$



Proof: Simulate NTM by 3-tape TM

- # nodes: $\leq b^{t(n)}$

where b = max # of nondet. choices NTM can make in any step.



Running time:

To simulate one root-to-node path:

$\leq O(t(n))$

Total time:

$b^{t(n)} \cdot O(t(n)) = 2^{t(n) \log b + O(\log t(n))}$

$= 2^{O(t(n))}$

Deterministic vs. nondeterministic time

Theorem: Let $t(n) \geq n$ be a function. Every NTM running in time $t(n)$ has an equivalent single-tape TM running in time $2^{O(t(n))}$

Proof: Simulate NTM by 3-tape TM in time $2^{O(t(n))}$

We know that a 3-tape TM can be simulated by a single-tape TM with quadratic overhead, hence we get running time

$= \left(2^{f(n)}\right)^2 = 2^{2f(n)}$

$$\left[2^{O(t(n))}\right]^2 = 2^{2 \cdot O(t(n))} = 2^{O(t(n))}$$

$2^{f(n)}$ for some $f(n) = O(n)$

Difference in time complexity

Extended Church-Turing Thesis:

At most polynomial difference in running time between all (reasonable) deterministic models

At most exponential difference in running time between deterministic and nondeterministic models

Nondeterministic time

Let $f : \mathbb{N} \rightarrow \mathbb{N}$

A NTM M runs in time $f(n)$ if on every input $w \in \Sigma^n$, M halts on w within at most $f(n)$ steps on every computational branch

$\mathrm{NTIME}(f(n))$ is a class (i.e., set) of languages:

A language $A \in \mathrm{NTIME}(f(n))$ if there exists an NTM M that

1) Decides A, and
2) Runs in time $O(f(n))$

NTIME explicitly

A language $A \in \mathrm{NTIME}(f(n))$ if there exists an NTM M such that, on every input $w \in \Sigma^*$

1. Every computational branch of M halts in either the accept or reject state within $f(|w|)$ steps

   M runs in nondet. time f(n).

2. If $w \in A$, then there exists an accepting computational branch of M on input w

3. If $w \notin A$, then every computational branch of M rejects on input w

M decides A

Complexity class NP

Definition: NP is the class of languages decidable in polynomial time on a nondeterministic TM

$$NP = \bigcup_{k=1}^{\infty} NTIME(n^k)$$

Reminder: $P = \bigcup_{k=1}^{\infty} TIME(n^k)$

$EXP = \bigcup_{k=1}^{\infty} TIME(2^{n^k})$

Which of the following are definitely true about NP?

a) $P \subseteq NP$ $\quad TIME(f(n)) \subseteq NTIME(f(n))$

b) $NP \subseteq P$

c) $NP \not\subseteq P$

??? win $1 million

d) $NP \subseteq EXP$ $\quad NTIME(f(n)) \subseteq TIME(2^{O(f(n))})$

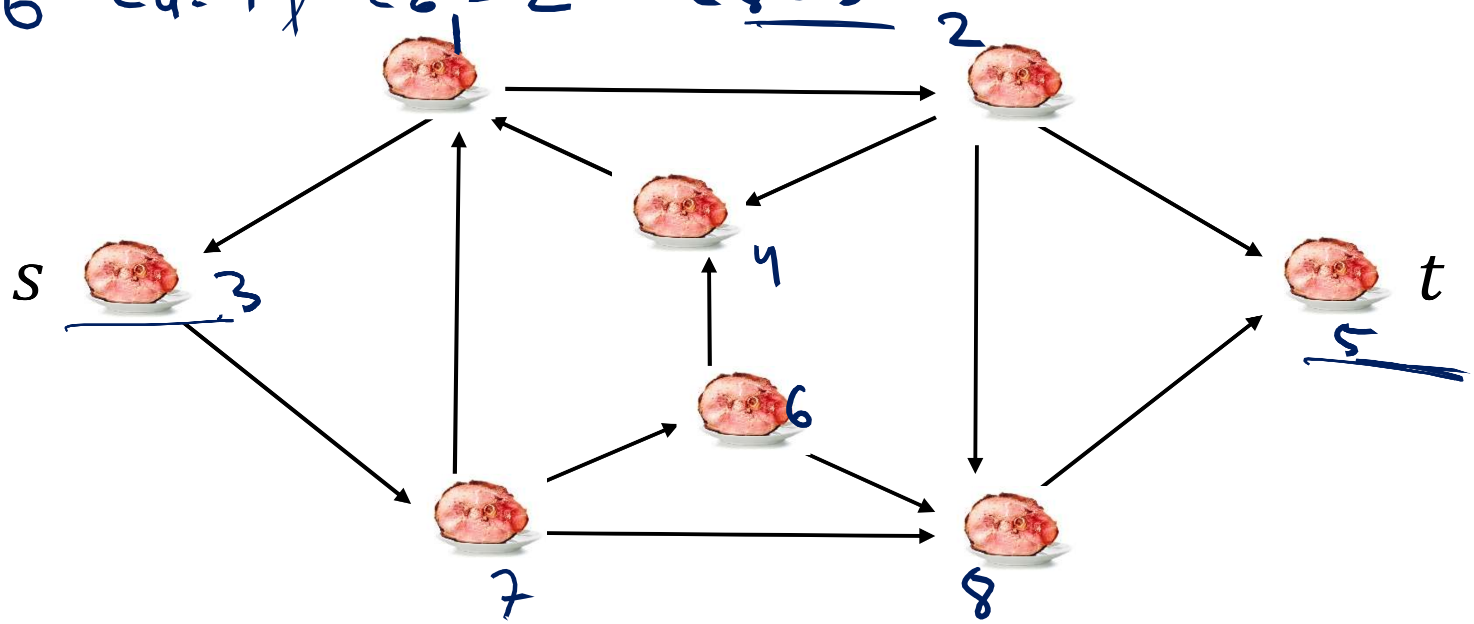
e) $EXP \subseteq NP$???

Hamiltonian Path

$HAMPATH = \{\langle G, s, t\rangle \mid G$ is a directed graph and there is a path from s to t that passes through every vertex exactly once$\}$

$c_1 = 3$ $c_2 = 7$, $c_5 = 1$ $c_7 = 8$

$c_3 = 6$ $c_4 = 4$ $c_6 = 2$ $c_8 = 5$



$HAMPATH \in \mathrm{NP}$

The following nondeterministic algorithm decides $HAMPATH$ in polynomial time:

$k = |V|$ Input length: k^2

On input $\langle G, s, t \rangle$: (Vertices of G are numbers $1, \dots, k$)

1. **Nondeterministically** guess a sequence $c_1, c_2, \dots, c_k$ of numbers $1, \dots, k$
2. Check that $c_1, c_2, \dots, c_k$ is a permutation: Every number $1, \dots, k$ appears exactly once
3. Check that $c_1 = s$, $c_k = t$, and there is an edge from every c_i to c_{i+1}
4. Accept if all checks pass, otherwise, reject.

Analyzing the algorithm

Need to check:

1) Correctness

If $\langle G,s,t\rangle \in HAMPATH$, $\exists$ a permutation $c_1,\ldots,c_n$ forming a path from s to t. So branch of computation on which c was guessed leads to accept.

If $\langle G,s,t\rangle \notin HAMPATH$, no sequence $c_1,\ldots,c_n$ represents a valid Ham path $\Rightarrow$ alg. rejects.

2) Running time

Step 1: $O(n \log n)$ (guess n $(\log n)$-bit numbers)

Step 2: $O((n \log n)^2)$ time (repeatedly scan)

Step 3: $O(\underbrace{(n \log n)}_{\text{iterations}} \cdot n^2)$ adj. matrix lookup

Total time: $O(n^3 \log n)$

polynomial in input length n^2.

An alternative characterization of NP

"Languages with polynomial-time verifiers"

How did we design an NTM for HAMPATH?

- Given a candidate path, it is easy (poly-time) to check whether this path is a Hamiltonian path
- We designed a poly-time NTM by nondeterministically guessing this path and then checking it
- Lots of problems have this structure (CLIQUE, 3-COLOR, COMPOSITE,…). They might be hard to solve, but a candidate solution is easy to check.

An alternative characterization of $\mathbf{NP}$

"Languages with polynomial-time verifiers"

A verifier for a language L is a deterministic algorithm V such that $w \in L$ iff there exists a string c such that $V(\langle w, c\rangle)$ accepts

Running time of a verifier is only measured in terms of $|w|$

V is a polynomial-time verifier if it runs in time polynomial in $|w|$ on every input $\langle w, c\rangle$

(Without loss of generality, $|c|$ is polynomial in $|w|$, i.e., $|c| = O(|w|^k)$ for some constant k)

$HAMPATH$ has a polynomial-time verifier

Certificate c:

Verifier V:

On input $\langle G, s, t; c\rangle$: (Vertices of G are numbers $1, \ldots, k$)

1. Check that $c_1, c_2, \ldots, c_k$ is a permutation: Every number $1, \ldots, k$ appears exactly once
2. Check that $c_1 = s$, $c_k = t$, and there is an edge from every c_i to c_{i+1}
3. Accept if all checks pass, otherwise, reject.

NP is the class of languages with polynomial-time verifiers

Theorem: A language $L \in \mathrm{NP}$ iff there is a polynomial-time verifier for L