BU CS 332 – Theory of Computation

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Lecture 22:

- P Examples
- NP

Reading:

Sipser Ch 7.2-7.3

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Complexity class P

Definition: P is the class of languages decidable in polynomial time on a basic single-tape (deterministic) TM

$$P = \bigcup_{k=1}^{\infty} \mathrm{TIME}(n^k)$$

- Class doesn't change if we substitute in another reasonable deterministic model (Extended Church-Turing)
- Cobham-Edmonds Thesis: Roughly captures class of problems that are feasible to solve on computers

Examples of languages in P

$PATH =$
$\{\langle G, s, t\rangle \mid G \text{ is a directed graph with a directed path from } s \text{ to } t\}$

Idea: Breadth-first search

Assume G presented as adjacency matrix

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"On input ⟨G, s, t⟩:
  1. Mark start vertex $s$
  2. For $i = 1, 2, \dots, |V|$:
  3.     Mark all neighbors of currently marked vertices
  4. If $t$ is marked, accept. Else, reject."
```

Examples of languages in P

$E_{\mathrm{DFA}} = \{\langle D\rangle \mid D \text{ is a DFA that recognizes the empty language}\}$

Examples of languages in P

- $RELPRIME = \{\langle x, y \rangle \mid x \text{ and } y \text{ are relatively prime}\}$

- $PRIMES = \{\langle x \rangle \mid x \text{ is prime}\}$

2006 Gödel Prize citation

The 2006 Gödel Prize for outstanding articles in theoretical computer science is awarded to Manindra Agrawal, Neeraj Kayal, and Nitin Saxena for their paper "PRIMES is in P."

In August 2002 one of the most ancient computational problems was finally solved....

A polynomial-time algorithm for $PRIMES$?

Consider the following algorithm for $PRIMES$

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On input ⟨$x$⟩:
For $b = 2, 3, 4, 5, \dots, \sqrt{x}$:
        - Try to divide $x$ by $b$
        - If $b$ divides $x$, reject
If all $b$ fail to divide $x$, accept
```

How many divisions does this algorithm require in terms of $n = |\langle x \rangle|$?   a) $O(\sqrt{n})$ b) $O(n)$ c) $2^{O(\sqrt{n})}$ d) $2^{O(n)}$

Beyond polynomial time

Definition: EXP is the class of languages decidable in exponential time on a basic single-tape (deterministic) TM

$$\mathrm{EXP} = \bigcup_{k=1}^{\infty} \mathrm{TIME}(2^{n^k})$$

Why study P ?

Criticism of the Cobham-Edmonds Thesis:

- Algorithms running in time n^{100} aren't really efficient

  Response: Runtimes improve with more research

- Does not capture some physically realizable models using randomness, quantum mechanics

  Response: Randomness may not change P, useful principles

$TIME(n)$ vs. $TIME(n^2)$

P vs. EXP

decidable vs. undecidable

Nondeterministic Time and NP

Extended Church-Turing Thesis

Every “reasonable” (physically realizable) model of computation can be simulated by a basic, single-tape TM with only a **polynomial** slowdown.

E.g., doubly infinite TMs, multi-tape TMs, RAM TMs

Does not include nondeterministic TMs (not reasonable)

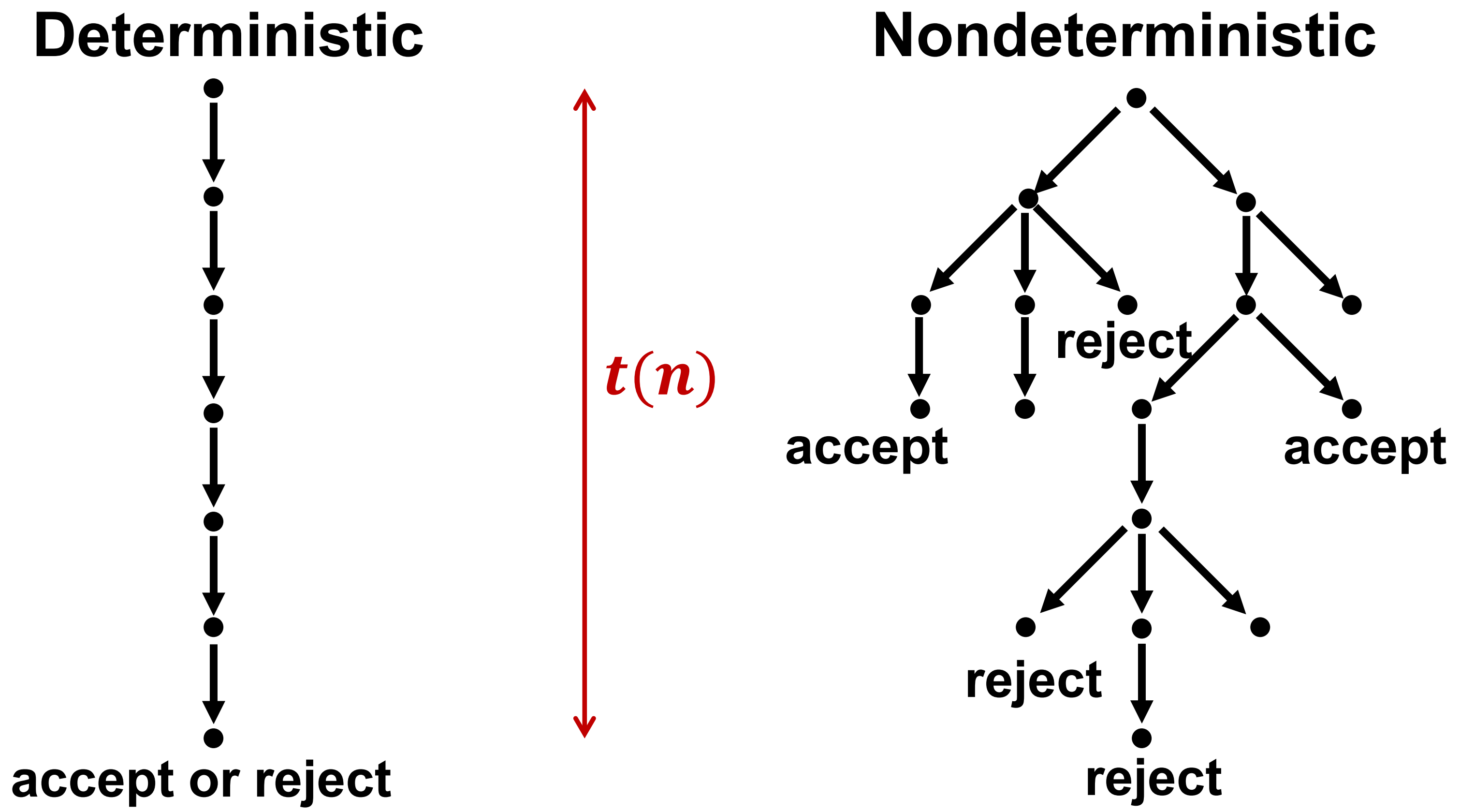
Nondeterministic time

Let $t\colon \mathbb{N} \to \mathbb{N}$

A NTM M runs in time $t(n)$ if on every input $w \in \Sigma^n$,

M halts on w within at most $t(n)$ steps on every computational branch

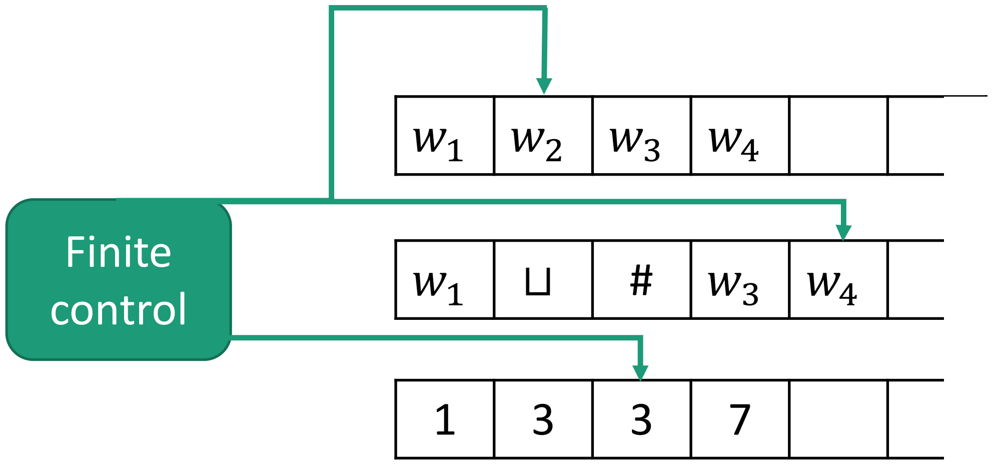
Deterministic vs. nondeterministic time



Deterministic vs. nondeterministic time

Theorem: Let $t(n) \geq n$ be a function. Every NTM running in time $t(n)$ has an equivalent single-tape TM running in time $2^{O(t(n))}$

Proof: Simulate NTM by 3-tape TM



Input w to N (read-only)

Simulation tape (run N on w using nondeterministic choices from tape 3)

Address in computation tree

Counting leaves

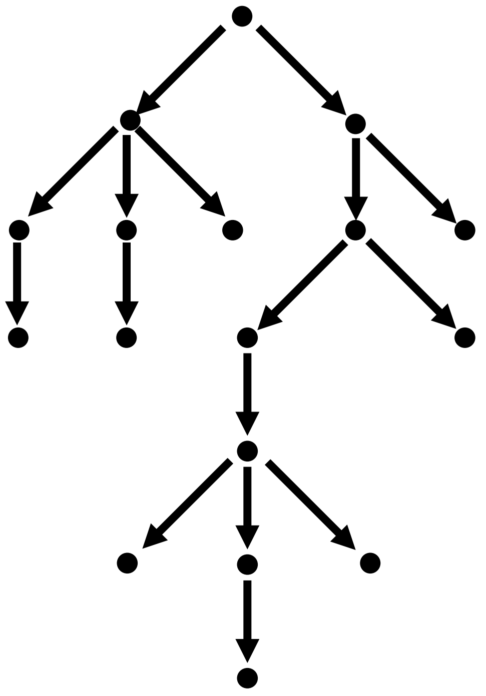
What is an upper bound on the maximum number of nodes in a tree with branching factor b and depth t?

a) bt

b) b^t

c) t^b

d) 2^t



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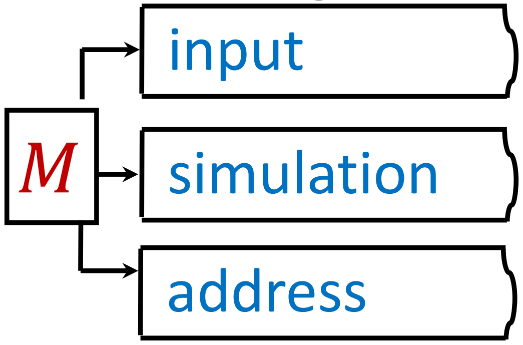


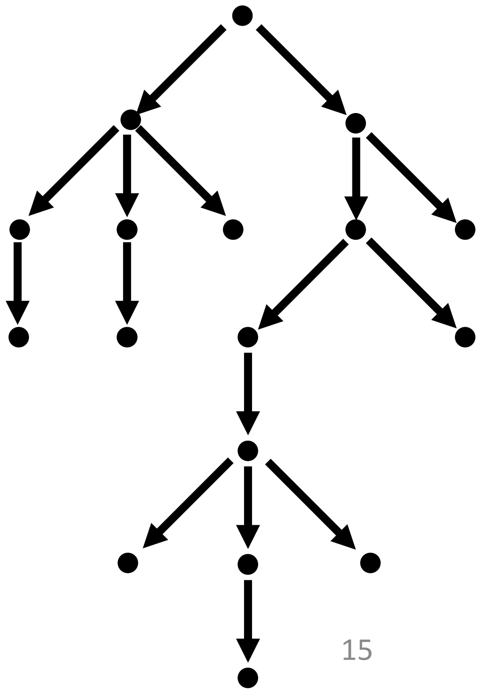
Proof: Simulate NTM by 3-tape TM

- # nodes:

<u>Running time:</u>

To simulate one root-to-node path:

Total time:



Deterministic vs. nondeterministic time

Theorem: Let $t(n) \geq n$ be a function. Every NTM running in time $t(n)$ has an equivalent single-tape TM running in time $2^{O(t(n))}$

Proof: Simulate NTM by 3-tape TM in time $2^{O(t(n))}$

We know that a 3-tape TM can be simulated by a single-tape TM with quadratic overhead, hence we get running time

$$\left(2^{O(t(n))}\right)^2 = 2^{2 \cdot O(t(n))} = 2^{O(t(n))}$$

Difference in time complexity

Extended Church-Turing Thesis:

At most polynomial difference in running time between all (reasonable) deterministic models

At most exponential difference in running time between deterministic and nondeterministic models

Nondeterministic time

Let $f : \mathbb{N} \to \mathbb{N}$

A NTM M runs in time $f(n)$ if on every input $w \in \Sigma^n$, M halts on w within at most $f(n)$ steps on every computational branch

$\mathrm{NTIME}(f(n))$ is a class (i.e., set) of languages:

A language $A \in \mathrm{NTIME}(f(n))$ if there exists an NTM M that

1) Decides A, and
2) Runs in time $O(f(n))$

NTIME explicitly

A language $A \in \mathrm{NTIME}(f(n))$ if there exists an NTM M such that, on every input $w \in \Sigma^*$

1. Every computational branch of M halts in either the accept or reject state within $f(|w|)$ steps

2. If $w \in A$, then there exists an accepting computational branch of M on input w

3. If $w \notin A$,then every computational branch of M rejects on input w

Definition: NP is the class of languages decidable in polynomial time on a nondeterministic TM

$\mathrm{NP} = \bigcup_{k=1}^{\infty} \mathrm{NTIME}(n^k)$

Which of the following are definitely true about NP?

a) $\mathrm{P} \subseteq \mathrm{NP}$

b) $\mathrm{NP} \subseteq \mathrm{P}$

c) $\mathrm{NP} \nsubseteq \mathrm{P}$

d) $\mathrm{NP} \subseteq \mathrm{EXP}$

e) $\mathrm{EXP} \subseteq \mathrm{NP}$

Complexity class NP

Definition: NP is the class of languages decidable in polynomial time on a nondeterministic TM

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Which of the following are definitely true about NP?

a) $\mathrm{P} \subseteq \mathrm{NP}$

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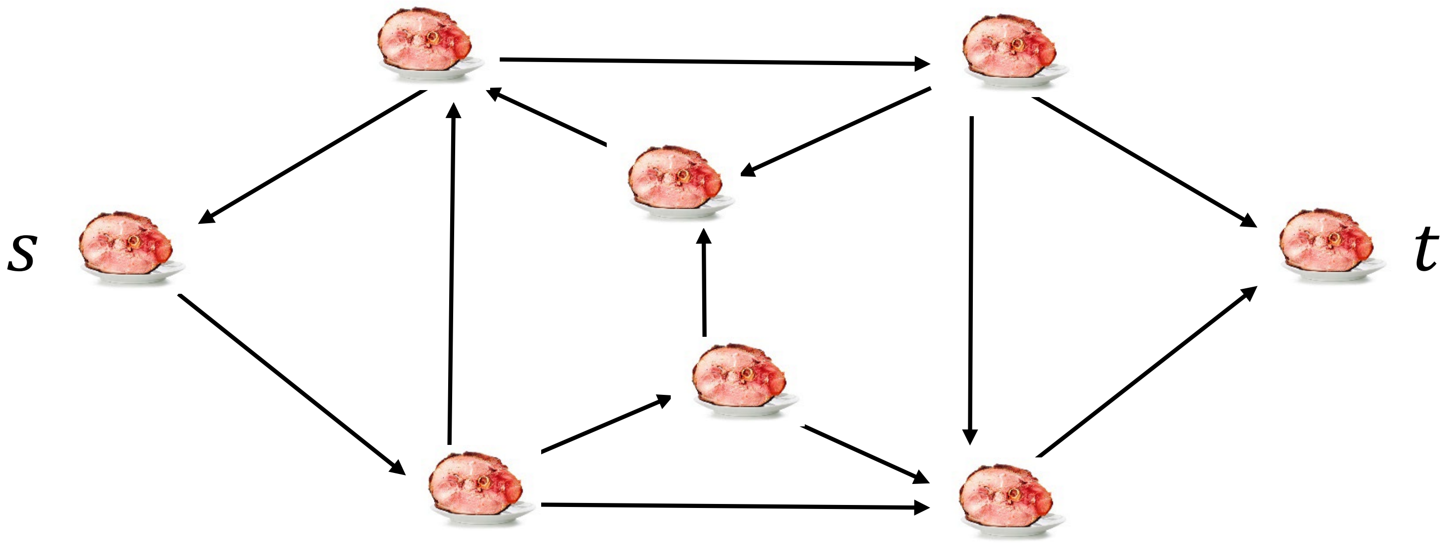
d) $\mathrm{NP} \subseteq \mathrm{EXP}$

e) $\mathrm{EXP} \subseteq \mathrm{NP}$

Hamiltonian Path

$HAMPATH = \{\langle G, s, t\rangle \mid G$ is a directed graph and there is a path from s to t that passes through every vertex exactly once$\}$



$HAMPATH \in \text{NP}$

The following nondeterministic algorithm decides $HAMPATH$ in polynomial time:

On input $\langle G, s, t\rangle$:   (Vertices of G are numbers $1, \ldots, k$)

1. **Nondeterministically** guess a sequence $c_1, c_2, \ldots, c_k$ of numbers $1, \ldots, k$
2. Check that $c_1, c_2, \ldots, c_k$ is a permutation: Every number $1, \ldots, k$ appears exactly once
3. Check that $c_1 = s$, $c_k = t$, and there is an edge from every c_i to c_{i+1}
4. Accept if all checks pass, otherwise, reject.

Analyzing the algorithm

Need to check:

1) Correctness

2) Running time

An alternative characterization of NP

"Languages with polynomial-time verifiers"

How did we design an NTM for HAMPATH?

- Given a candidate path, it is easy (poly-time) to check whether this path is a Hamiltonian path
- We designed a poly-time NTM by nondeterministically guessing this path and then checking it
- Lots of problems have this structure (CLIQUE, 3-COLOR, COMPOSITE,…). They might be hard to solve, but a candidate solution is easy to check.

An alternative characterization of NP

"Languages with polynomial-time verifiers"

A verifier for a language L is a deterministic algorithm V such that $w \in L$ iff there exists a string c such that $V(\langle w, c \rangle)$ accepts

Running time of a verifier is only measured in terms of $|w|$

V is a polynomial-time verifier if it runs in time polynomial in $|w|$ on every input $\langle w, c \rangle$

(Without loss of generality, $|c|$ is polynomial in $|w|$, i.e., $|c| = O(|w|^k)$ for some constant k)

$HAMPATH$ has a polynomial-time verifier

Certificate c:

Verifier V:

On input $\langle G, s, t; c \rangle$: (Vertices of G are numbers $1, \ldots, k$)

1. Check that $c_1, c_2, \ldots, c_k$ is a permutation: Every number $1, \ldots, k$ appears exactly once
2. Check that $c_1 = s$, $c_k = t$, and there is an edge from every c_i to c_{i+1}
3. Accept if all checks pass, otherwise, reject.

NP is the class of languages with polynomial-time verifiers

Theorem: A language $L \in \mathrm{NP}$ iff there is a polynomial-time verifier for L