BU CS 332 – Theory of Computation

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Lecture 23:

- NP-completeness

Reading:

Sipser Ch 7.4-7.5

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Last time: Two equivalent definitions of NP

1) NP is the class of languages decidable in polynomial time on a nondeterministic TM

$$\mathrm{NP} = \bigcup_{k=1}^{\infty} \mathrm{NTIME}(n^k)$$

2) A polynomial-time verifier for a language L is a deterministic $\mathrm{poly}(|w|)$-time algorithm V such that

$w \in L \iff$ there exists a certificate c such that $V(\langle w, c \rangle)$ accepts

Theorem: A language $L \in \mathrm{NP}$ iff there is a polynomial-time verifier for L

Examples of NP languages

- Hamiltonian path

  Given a graph G and vertices s, t, does G contain a Hamiltonian path from s to t?

- Clique

  Given a graph G and natural number k, does G contain a clique of size k?

- Subset Sum

  Given a list of natural numbers $x_1, \dots, x_k, t$ is there a subset of the numbers $x_1, \dots, x_k$ that sum up to exactly t?

- Boolean satisfiability (SAT)

  Given a Boolean formula, is there a satisfying assignment?

- Vertex Cover

  Given a graph G and natural number k, does G contain a vertex cover of size k?

- Traveling Salesperson

NP-Completeness

Understanding the $\mathbf{P}$ vs. $\mathbf{NP}$ question

Most believe $\mathrm{P} \neq \mathrm{NP}$, but we are very far from proving it

Question 1: How can studying specific computational problems help us get a handle on resolving P vs. NP?

Question 2: What would $\mathrm{P} \neq \mathrm{NP}$ allow us to conclude about specific problems we care about?

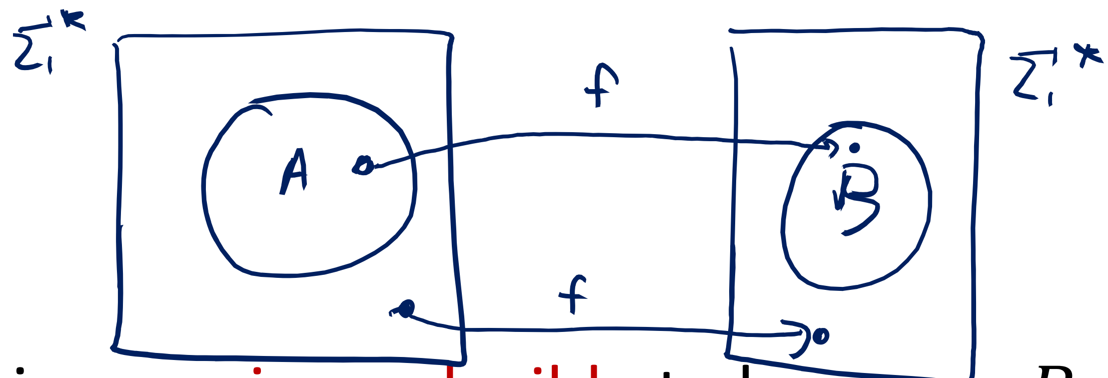
Idea: Identify the "hardest" problems in NP

Languages $L \in \mathrm{NP}$ such that $L \in \mathrm{P}$ iff $\mathrm{P} = \mathrm{NP}$

Recall: Mapping reducibility

Definition:

A function $f: \Sigma^* \to \Sigma^*$ is computable if there is a TM M which, given as input any $w \in \Sigma^*$, halts with only $f(w)$ on its tape.



Definition:

Language A is mapping reducible to language B, written

$$A \leq_{\mathrm{m}} B$$

if there is a computable function $f: \Sigma^* \to \Sigma^*$ such that for all strings $w \in \Sigma^*$, we have $w \in A \Leftrightarrow f(w) \in B$

Polynomial-time reducibility

Definition:

A function $f: \Sigma^* \rightarrow \Sigma^*$ is polynomial-time computable if there is a polynomial-time TM M which, given as input any $w \in \Sigma^*$, halts with only $f(w)$ on its tape.

Definition:

Language A is polynomial-time reducible to language B, written

$$A \leq_{\mathrm{p}} B$$

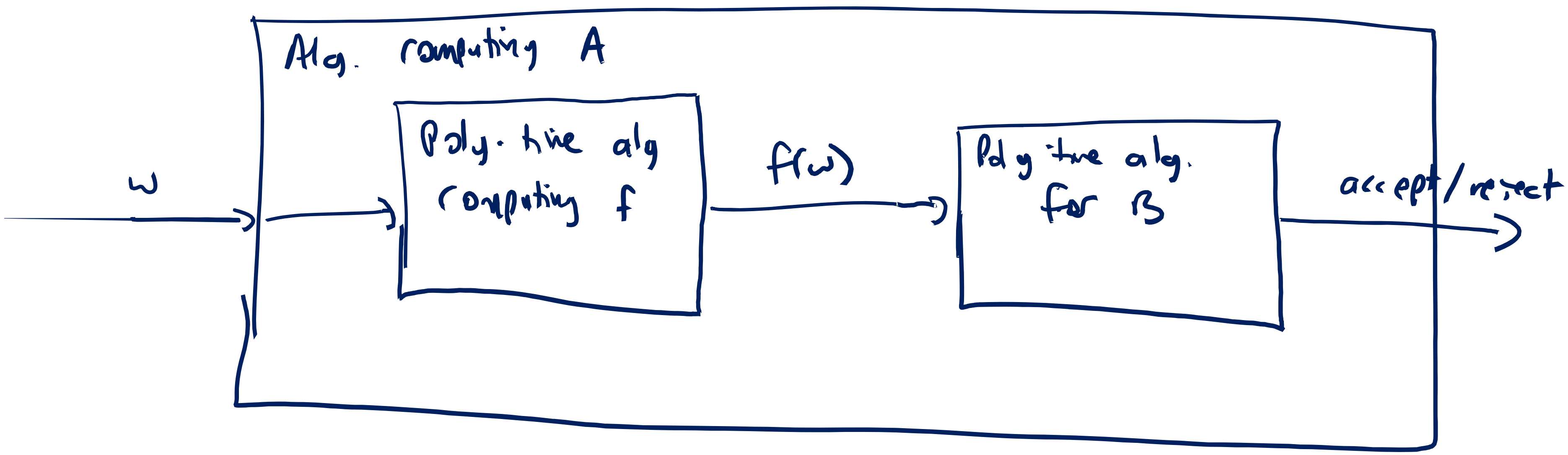
if there is a polynomial-time computable function $f: \Sigma^* \rightarrow \Sigma^*$ such that for all strings $w \in \Sigma^*$, we have $w \in A \Leftrightarrow f(w) \in B$

Implications of poly-time reducibility

cf. If $A \le_m B$ and B is decidable, then A decidable

Theorem: If $A \le_{\mathrm{p}} B$ and $B \in \mathrm{P}$, then $A \in \mathrm{P}$

Proof: Let M decide B in poly time, and let f be a poly-time reduction from A to B. The following TM decides A in poly time:



Thm: If $A \le_p B$ and $B \in P$, then $A \in P$

Proof. Let M decide B in poly-time

Consider the following TM N:

On input w:

1) Compute $f(w)$

2) Run M on input $f(w)$. If accepts, accept.
If rejects, reject.

Correctness:

$w \in A \iff f(w) \in B$ (def. of poly-time reduction)

$\iff M$ accepts $f(w)$ [M decides B]

$\iff N$ accepts w [by construction of alg. N]

Run time:

Computing $f(w)$ takes poly($|w|$) time
[by def. of poly-time reduction]

Length $|f(w)| =$ poly($|w|$)

$\Rightarrow M(f(w))$ takes poly(poly($|w|$)) time, which is polynomial.

Is NP closed under poly-time reductions?

If $A \leq_{\mathrm{p}} B$ and B is in NP, does that mean A is also in NP?

Analogous to how $A \leq_m B$ and B recognizable, then A is recognizable

a) Yes, the same proof works using NTMs instead of TMs

b) No, because the new machine is an NTM instead of a deterministic TM

c) No, because the new NTM may not run in polynomial time

d) No, because the new NTM may accept some inputs it should reject

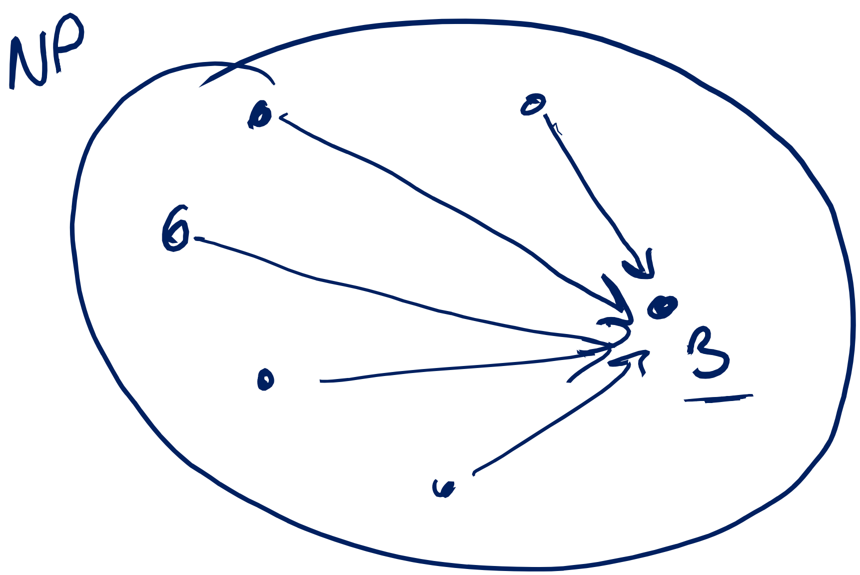
e) No, because the new NTM may reject some inputs it should accept

NP-completeness

Definition: A language B is NP-complete if

1) $B \in \mathrm{NP}$, and

2) B is NP-hard: Every language $A \in \mathrm{NP}$ is poly-time reducible to B, i.e., $A \leq_{\mathrm{p}} B$



$A \to B$ means $A \leq_p B$

Implications of NP-completeness

Theorem: Suppose B is NP-complete.

Then $B \in \mathrm{P}$ iff $\mathrm{P} = \mathrm{NP}$

Proof:

$\Leftarrow$) WTS if $P = NP$ then $B \in P$

Why? B NP-complete $\Rightarrow B \in NP = P$. ✓

$\Rightarrow$) WTS if $B \in P$ then $P = NP$

Let $A \in NP$ be any language.

Since B is NP-hard, $A \leq_p B \in P$

$\Rightarrow$ By Thm from slide 8, $A \in P$.

$\Rightarrow NP \subseteq P$. Already know $P \subseteq NP \Rightarrow P = NP$.

Implications of NP-completeness

Theorem: Suppose B is NP-complete.

Then $B \in \mathrm{P}$ iff $\mathrm{P} = \mathrm{NP}$

Consequences of B being NP-complete:

1) If you want to prove $\mathrm{P} = \mathrm{NP}$, you just have to prove $B \in \mathrm{P}$
2) If you want to prove $\mathrm{P} \neq \mathrm{NP}$, a good candidate is to try to show that $B \notin \mathrm{P}$
3) If you believe $\mathrm{P} \neq \mathrm{NP}$, then you also believe $B \notin \mathrm{P}$

Cook-Levin Theorem and NP-Complete Problems

Do NP-complete problems exist?

t encoded in unary

Theorem: $TMSAT = \{\langle N, w, \boxed{1^t} \rangle \mid$ NTM N accepts input w within t steps$\}$ is NP-complete

Proof sketch: 1) $TMSAT \in \mathrm{NP}$: Certificate = t nondeterministic guesses made by N, verifier checks that N accepts w within t steps under those guesses.

2) $TMSAT$ is NP-hard: Let $L \in \mathrm{NP}$ be decided by NTM N running in time $T(n)$. The following poly-time TM shows $L \leq_{\mathrm{p}} TMSAT$:

"On input w (an instance of L):

Output $\langle N, w, 1^{T(|w|)} \rangle$."

Correctness:

$w \in L \iff$ NTM N accepts w w/in $T(|w|)$ steps on some branch

$\iff \langle N, w, 1^{T(|w|)} \rangle \in TMSAT$

Cook-Levin Theorem

Theorem: SAT (Boolean satisfiability) is NP-complete

"Proof": Already know $SAT \in \mathrm{NP}$. (Much) harder direction: Need to show every problem in NP reduces to SAT

Stephen A. Cook (1971)

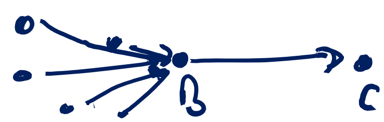
Leonid Levin (1973)

New NP-complete problems from old

Lemma: If $A \overset{g}{\leq_{\mathrm{p}}} B$ and $B \overset{f}{\leq_{\mathrm{p}}} C$, then $A \overset{f \circ g}{\leq_{\mathrm{p}}} C$

(poly-time reducibility is <u>transitive</u>)

Theorem: If $B \leq_{\mathrm{p}} C$ for some NP-hard language B, then C is also NP-hard

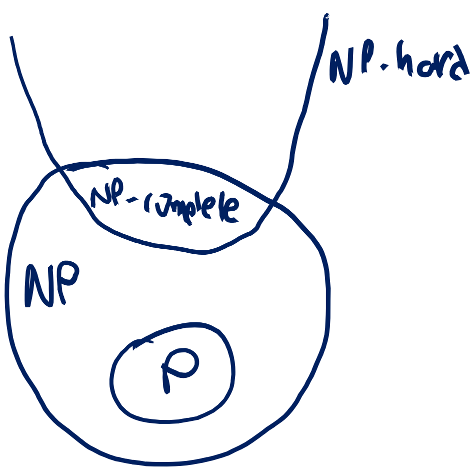


$\forall\ A \in NP,\ A \leq_p B$ [defn]

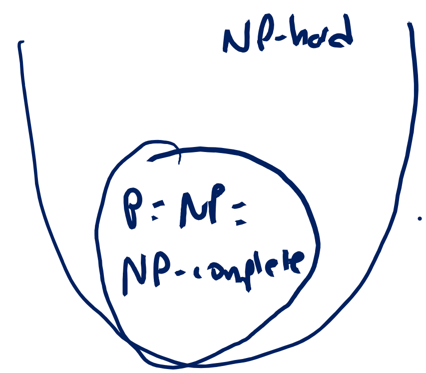
$\Rightarrow A \leq_p C$ [transitivity]

Corollary: If $C \in \mathrm{NP}$ and $B \leq_{\mathrm{p}} C$ for some NP-complete language B, then C is also NP-complete

$P \neq NP$

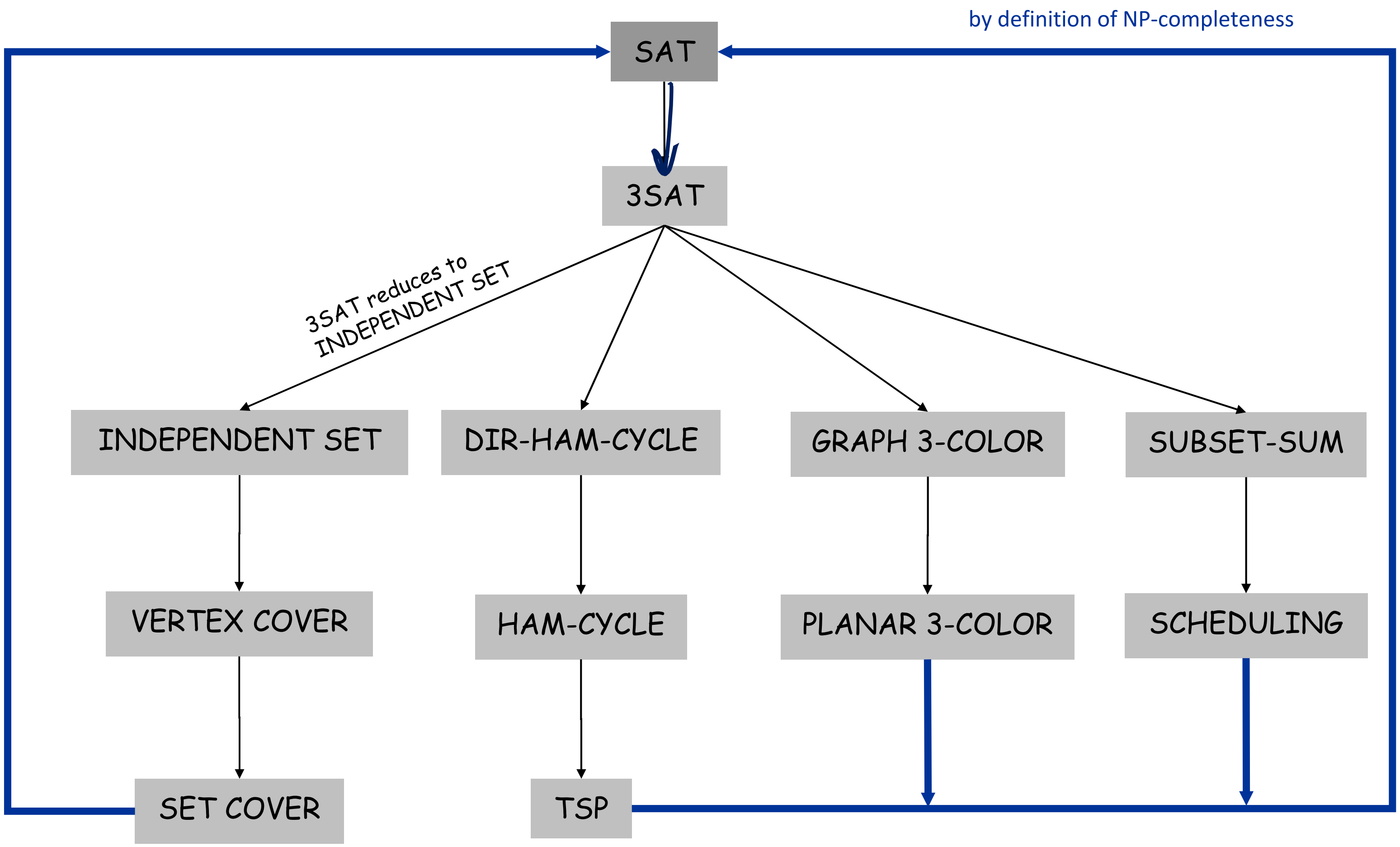


$P = NP$



New NP-complete problems from old

All problems below are NP-complete and hence poly-time reduce to one another!



$3SAT$ (3-CNF Satisfiability)

Definitions:

- A literal either a variable ~~of~~ or its negation x_5 , $\overline{x_7}$
- A clause is a disjunction (OR) of literals Ex. $x_5 \vee \overline{x_7} \vee x_2$
- A 3-CNF is a conjunction (AND) of clauses where each clause contains exactly 3 literals

Ex. $C_1 \wedge C_2 \wedge \ldots \wedge C_m =$

$$(x_5 \vee \overline{x_7} \vee x_2) \wedge (\overline{x_3} \vee x_4 \vee x_1) \wedge \cdots \wedge (x_1 \vee x_1 \vee x_1)$$

$$3SAT = \{\langle\varphi\rangle | \varphi \text{ is a satisfiable } 3-\text{CNF}\}$$

$3SAT$ is NP-complete

Theorem: $3SAT$ is NP-complete

Proof idea: 1) $3SAT$ is in NP (why?)

2) Show that $\underbrace{SAT}_{\text{know is NP-complete}} \leq_p 3SAT$ shows $3SAT$ is NP-hard

Your classmate suggests the following reduction from SAT to $3SAT$: "On input φ, a 3-CNF formula (an instance of $3SAT$), output φ, which is already an instance of SAT." Is this reduction correct?

a) Yes, this is a poly-time reduction from SAT to $3SAT$

b) No, because φ is not an instance of the SAT problem

c) No, the reduction does not run in poly time

d) No, this is a reduction from $3SAT$ to SAT; it goes in the wrong direction

$3SAT$ is NP-complete

Theorem: $3SAT$ is NP-complete

Proof idea: 1) $3SAT$ is in NP (why?)

2) Show that $SAT \leq_{\mathrm{p}} 3SAT$

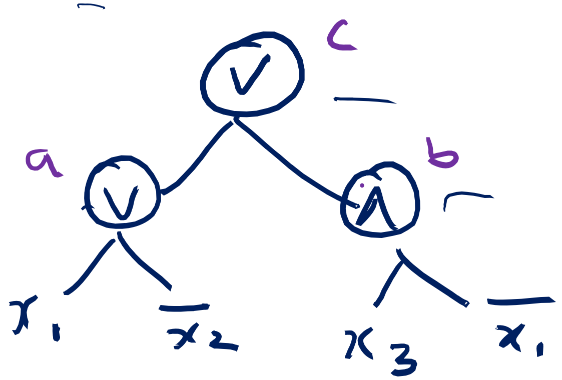
Idea of reduction: Give a poly-time algorithm converting an arbitrary formula φ into a 3CNF ψ such that φ is satisfiable iff ψ is satisfiable

Formal reduction not given here

Illustration of conversion from φ to ψ

instance of SAT
arbitrary formula

Step 1: Push all negations in φ to variable level



Step 2: $\underbrace{(a = x_1 \vee \overline{x_2})}_{\text{"pseudo clauses"}} \wedge (b = x_3 \wedge \overline{x_1}) \wedge (c = a \vee b) \wedge (c = 1)$

Step 3: Convert each $(a = x \vee y)$ into a 8-clause 3CNF

Possible because every function $f: \{0,1\}^3 \to \{0,1\}$ has a 3CNF representation

Some general reduction strategies

- Reduction by simple equivalence

  Ex. $IND - SET \leq_{\mathrm{p}} VERTEX - COVER$

  $VERTEX - COVER \leq_{\mathrm{p}} IND - SET$

- Reduction from special case to general case

  Ex. $VERTEX - COVER \leq_{\mathrm{p}} SET - COVER$

  $3SAT \leq_{\mathrm{p}} SAT$

- “Gadget” reductions

  Ex. $SAT \leq_{\mathrm{p}} 3SAT$

  $3SAT \leq_{\mathrm{p}} IND - SET$