Professor Mark Bun
March 23, 2020

Homework 6 – Due Monday, March 30, 2020 before 2:00PM

Reminder Collaboration is permitted, but you must write the solutions *by yourself without assistance*, and be ready to explain them orally to the course staff if asked. You must also identify your collaborators. Getting solutions from outside sources such as the Web or students not enrolled in the class is strictly forbidden.

Exercises Please practice on the following exercises and solved problems in Chapter 5: 5.1, 5.2, 5.4–5.7, 5.10, 5.11. The material they cover may appear on exams.

Note If you need to describe a Turing machine, please give a high-level description as in Chapter 4.1 of Sipser or in Lecture 13.

Problems There are 3 mandatory problems and one bonus problem.

1. (**Adding TM**) A TM *correctly adds* if, given two binary numbers, separated by #, it halts with their sum (in binary) on its tape. (It does not matter what it does on other inputs.) Consider the problem of determining whether a TM *correctly adds*. Formulate this problem as a language and prove it is undecidable.

2. (**Enthusiastic TM**) Consider the problem of determining whether a given TM ever[1] writes "332" on three adjacent squares of its tape. You may assume that the input alphabet of this TM is $\{0, 1\}$ and the tape alphabet is $\{0, 1, 2, \ldots, 9\}$.

   (a) Formulate this problem as a language $ENTHUSIASTIC_{\mathrm{TM}}$.

   (b) Show $ENTHUSIASTIC_{\mathrm{TM}}$ is undecidable.

   (c) Prove that $ENTHUSIASTIC_{\mathrm{TM}}$ is Turing-recognizable.

   (d) Is $\overline{ENTHUSIASTIC_{\mathrm{TM}}}$ Turing-recognizable? Prove or disprove.

3. (**Recognizable and unrecognizable languages)**

   (a) **(Complement of** $\mathrm{ALL}_{\mathsf{CFG}}$**)** Let $ALL_{\mathsf{CFG}} = \{\langle G\rangle \mid G \text{ is a } \mathsf{CFG} \text{ with terminal set } \Sigma \text{ and } L(G) = \Sigma^*\}$. Prove that the complement of ALL_{CFG} is Turing-recognizable.

   (b) (**Accepting its own description**) Consider the self-acceptance problem for Turing machines described in Lecture 15: $SA_{\mathsf{TM}} = \{\langle M\rangle \mid M \text{ is a } \mathsf{TM} \text{ that accepts on input } \langle M\rangle\}$. Modify the diagonalization proof of undecidability for $\overline{SA_{\mathsf{TM}}}$ to show that $\overline{SA_{\mathsf{TM}}}$ is not even Turing-recognizable (i.e., SA_{TM} is not co-Turing-recognizable).

[1] on some input

(c) (**DECIDER$_{\mathsf{TM}}$**) Let $\text{DECIDER}_{\mathsf{TM}} = \{\langle M \rangle \mid M \text{ is a } \mathsf{TM} \text{ that halts on every input}\}$. Prove the following statements.

i. $\overline{\text{DECIDER}_{\mathsf{TM}}}$ is not Turing-recognizable (i.e., $\text{DECIDER}_{\mathsf{TM}}$ is not co-Turing-recognizable).

ii. $\text{DECIDER}_{\mathsf{TM}}$ is not Turing-recognizable.

4* (**Bonus, no collaboration is allowed**) Let A be a Turing-recognizable language consisting of descriptions of Turing machines, $\{\langle M_1 \rangle, \langle M_2 \rangle, ...\}$, where every M_i is a decider. Prove that some decidable language D is not decided by any decider M_i whose description appears in A. (*Hint:* Use diagonalization and consider an enumerator for A.)