BU CS 332 – Theory of Computation

Lecture 2:

- Deterministic Finite Automata
- Regular Operations
- Non-deterministic FAs

Reading:

Sipser Ch 1.1-1.2

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Deterministic Finite Automata

A (Real-Life?) Example

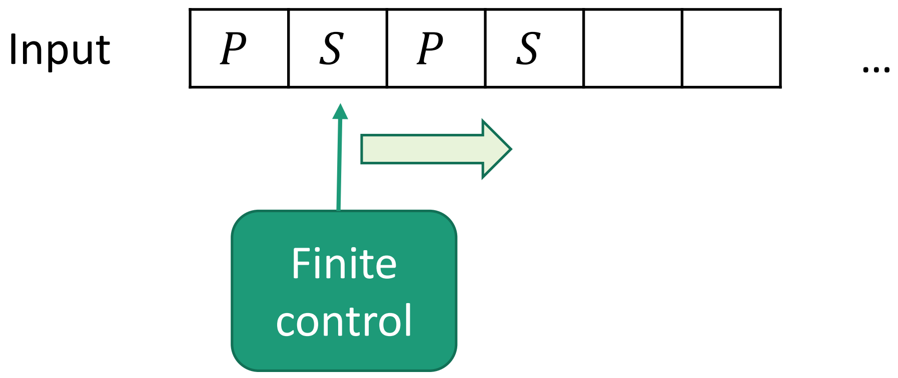
- Example: Car stereo
- P = Power button (ON/OFF)
- S = Source button (cycles through Radio/Bluetooth/USB)

  Only works when stereo is ON, but source remembered when stereo is OFF
- Starts OFF in Radio mode

- A computational problem: Does a sequence of button presses in $\{P, S\}^*$ leave the stereo ON in USB mode?

Machine Models

- Finite Automata (FAs): Machine with a finite amount of unstructured memory



Control scans left-to-right

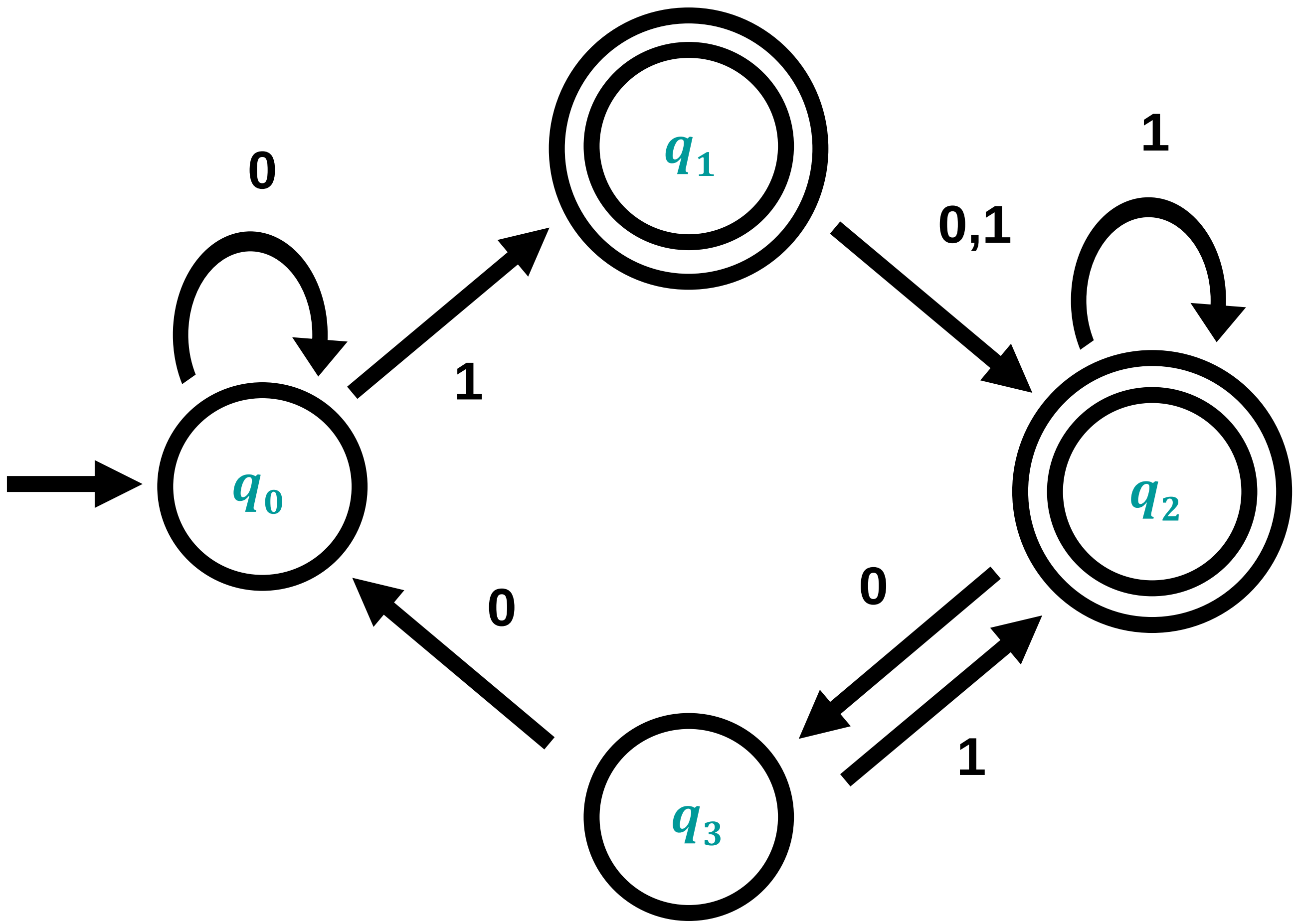
A DFA for the car stereo problem

A DFA for Parity

Parity: Given a string consisting of a's and b's, does it contain an even number of a's?

$\Sigma = \{a, b\}$ $\quad$ $L = \{w \mid w \text{ contains an even number of } a\text{'s}\}$

Anatomy of a DFA



Formal Definition of a DFA

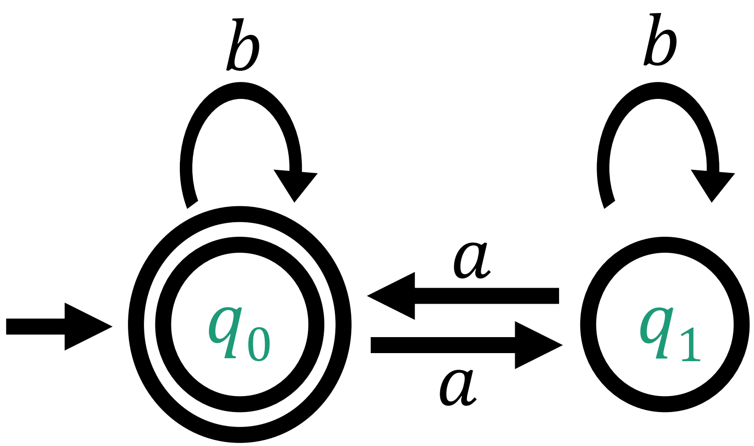
A finite automaton is a 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$

- Q is the set of states
- Σ is the alphabet
- $\delta : Q \times \Sigma \rightarrow Q$ is the transition function
- $q_0 \in Q$ is the start state
- $F \subseteq Q$ is the set of accept states

A DFA for Parity

Parity: Given a string consisting of a's and b's, does it contain an even number of a's?

$\Sigma = \{a, b\}$ $\quad$ $L = \{w \mid w$ contains an even number of a's$\}$



State set Q =

Alphabet Σ =

Transition function δ

δ	a	b
q_0		
q_1		

Start state q_0

Set of accept states F =

Formal Definition of DFA Computation

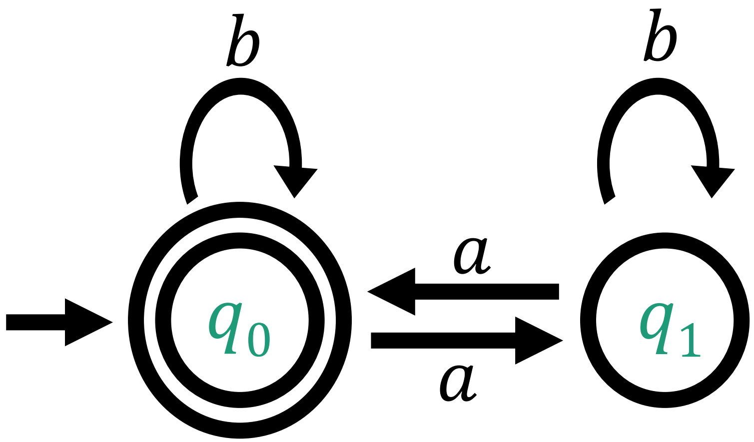
A DFA $M = (Q, \Sigma, \delta, q_0, F)$ accepts a string $w = w_1 w_2 \cdots w_n \in \Sigma^*$ (where each $w_i \in \Sigma$) if there exist $r_0, \ldots, r_n \in Q$ such that

1. $r_0 = q_0$
2. $\delta(r_i, w_{i+1}) = r_{i+1}$ for each $i = 0, \ldots, n-1$, and
3. $r_n \in F$

$L(M)$ = the language of machine M
= set of all (finite) strings machine M accepts

M recognizes the language $L(M)$

Example: Computing with the Parity DFA



Let $w = abba$

Does M accept w?

A DFA $M = (Q, \Sigma, \delta, q_0, F)$ accepts a string $w = w_1 w_2 \cdots w_n \in \Sigma^*$ (where each $w_i \in \Sigma$) if there exist $r_0, \ldots, r_n \in Q$ such that

1. $r_0 = q_0$
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3. $r_n \in F$

Automata Tutor

http://automatatutor.com/

Regular Languages

<u>Definition:</u> A language is regular if it is recognized by a DFA

$L = \{ w \in \{a, b\}^* \mid w$ **has an even number of** a**'s** $\}$ **is regular**

$L = \{ w \in \{0, 1\}^* \mid w$ **contains** 001 $\}$ **is regular**

Many interesting programs recognize regular languages

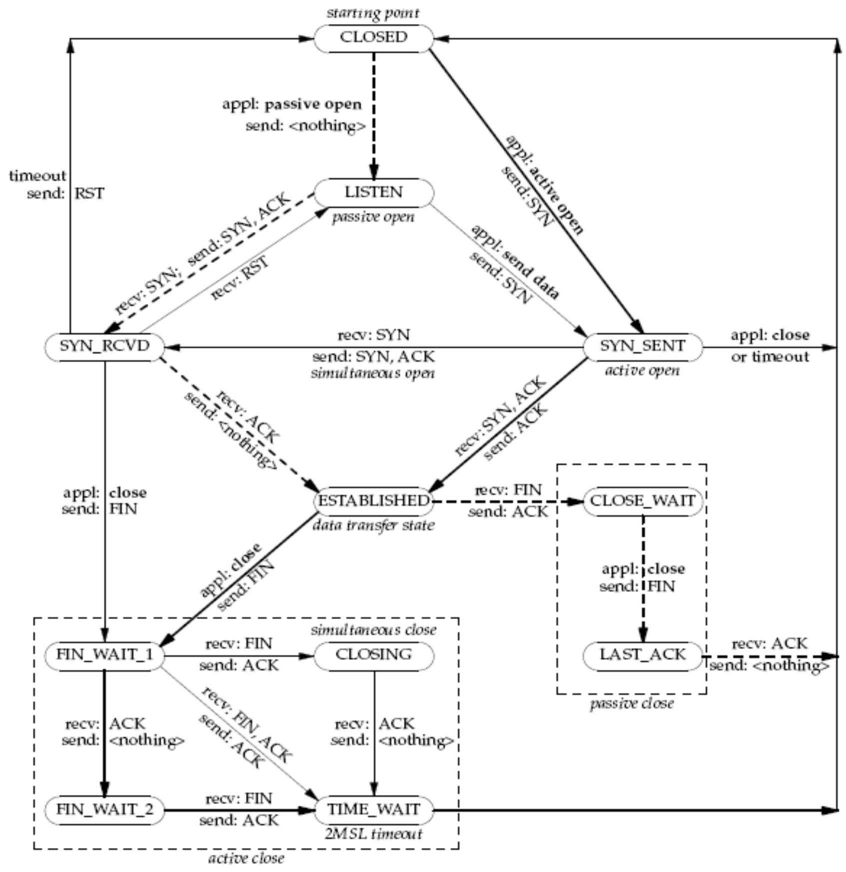
NETWORK PROTOCOLS

COMPILERS

GENETIC TESTING

ARITHMETIC

Internet Transmission Control Protocol



Let TCPS = { w | w is a complete TCP Session}

Theorem. TCPS is regular

Compilers

Comments :

Are delimited by /* */

Cannot have nested /* */

Must be closed by */

***/ is illegal outside a comment**

COMMENTS = {strings over {0,1, /, *} with legal comments}

Theorem. COMMENTS is regular.

Genetic Testing

DNA sequences are strings over the alphabet $\{A, C, G, T\}$.

A gene g is a special substring over this alphabet.

A genetic test searches a DNA sequence for a gene.

GENETICTEST_g = {strings over $\{A, C, G, T\}$ containing g as a substring}

Theorem. GENETICTEST_g is regular for every gene g.

Arithmetic

LET $\Sigma_3 = \left\{ \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\}$

- **A string over Σ_3 has three ROWS (ROW$_1$, ROW$_2$, ROW$_3$)**
- **Each ROW $b_0 b_1 b_2 \ldots b_N$ represents the integer**

$$b_0 + 2b_1 + \ldots + 2^N b_N.$$

- **Let ADD = $\{S \in \Sigma_3^* \mid \text{ROW}_1 + \text{ROW}_2 = \text{ROW}_3\}$**

Theorem. **ADD is regular.**

Regular Operations

An Analogy

In algebra, we try to identify operations which are common to many different mathematical structures

Example: The integers $\mathbb{Z} = \{\ldots -2, -1, 0, 1, 2, \ldots\}$ are **closed** under

- Addition: $x + y$
- Multiplication: $x \times y$
- Negation: $-x$
- …but NOT Division: $x \,/\, y$

We'd like to investigate similar closure properties of the class of regular languages

Regular operations on languages

Let $A, B \subseteq \Sigma^*$ be languages. Define

Union: $A \cup B =$

Concatenation: $A \circ B =$

Star: $A^* =$

Other operations

Let $A, B \subseteq \Sigma^*$ be languages. Define

Complement: $\bar{A} =$

Intersection: $A \cap B =$

Reverse: $A^R =$

Closure properties of the regular languages

Theorem: The class of regular languages is closed under all three regular operations (union, concatenation, star), as well as under complement, intersection, and reverse.

i.e., if A and B are regular, applying any of these operations yields a regular language

Proving Closure Properties

Complement

Complement: $\bar{A} = \{ w \mid w \notin A \}$

Theorem: If A is regular, then $\bar{A}$ is also regular

Proof idea:

Union

Union: $A \cup B = \{ w \mid w \in A \text{ or } w \in B \}$

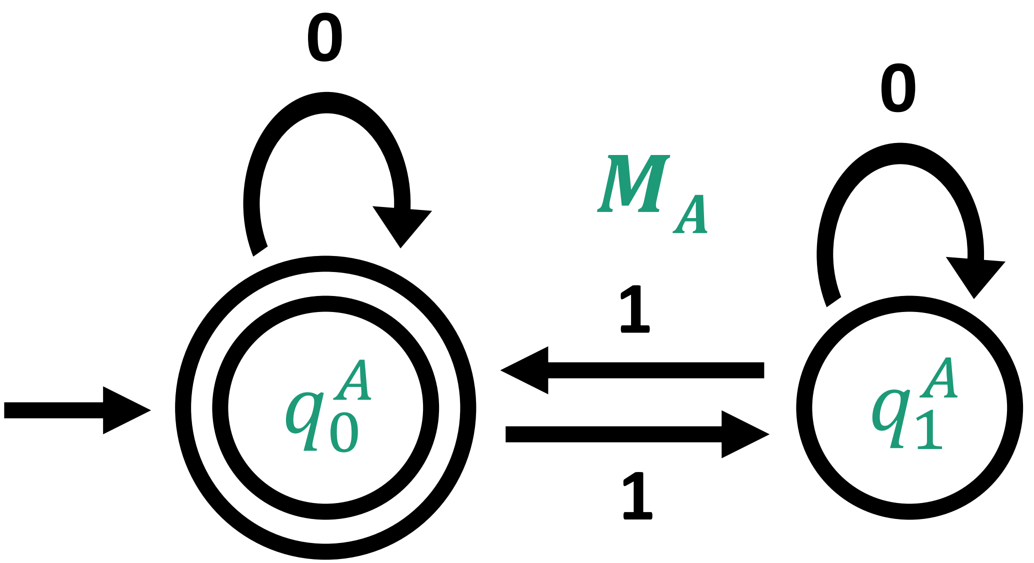
Theorem: If A and B are regular, then so is $A \cup B$

Proof:

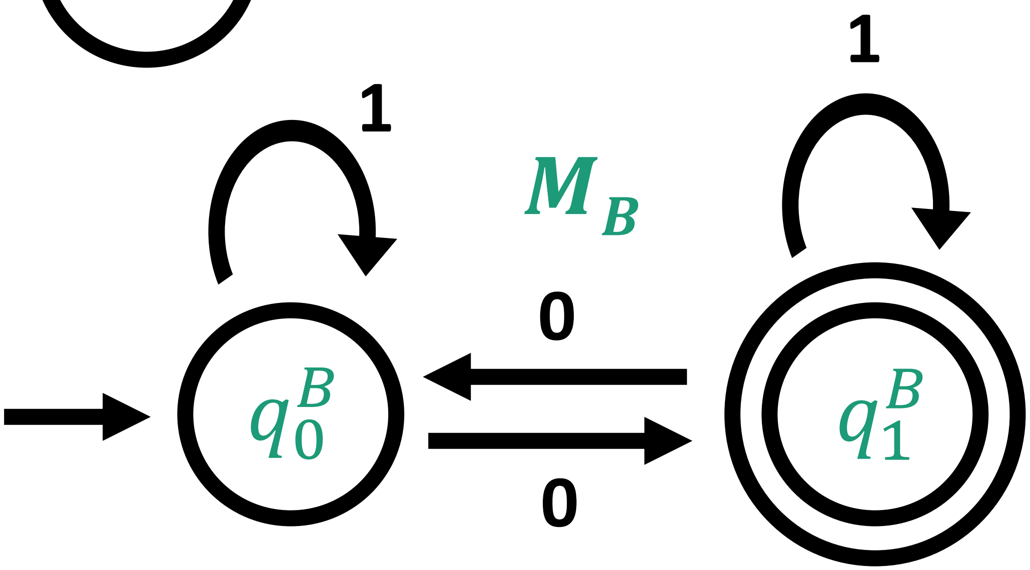
Let $M_A = (Q_A, \Sigma, \delta_A, q_0^A, F_A)$ be a DFA recognizing A and
$M_B = (Q_B, \Sigma, \delta_B, q_0^B, F_B)$ be a DFA recognizing B

<u>Goal:</u> Construct a DFA $M = (Q, \Sigma, \delta, q_0, F)$
that recognizes $A \cup B$

Example



$M = ?$



Closure under union proof (cont'd)

Idea: Run both M_A and M_B at the same time

"Cross-product construction"

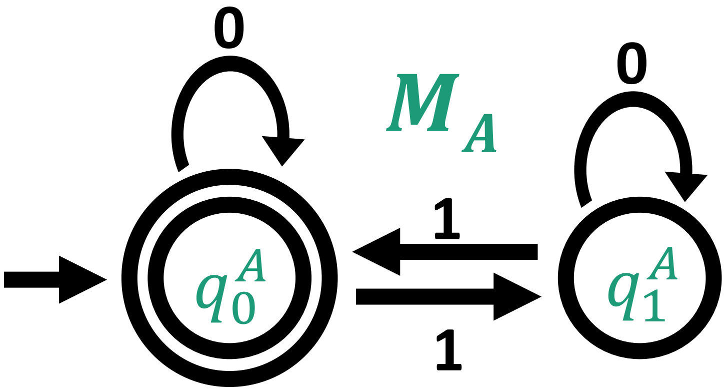
$$Q = Q_A \times Q_B$$
$$= \{(q_A, q_B) \mid q_A \in A \text{ and } q_B \in B\}$$

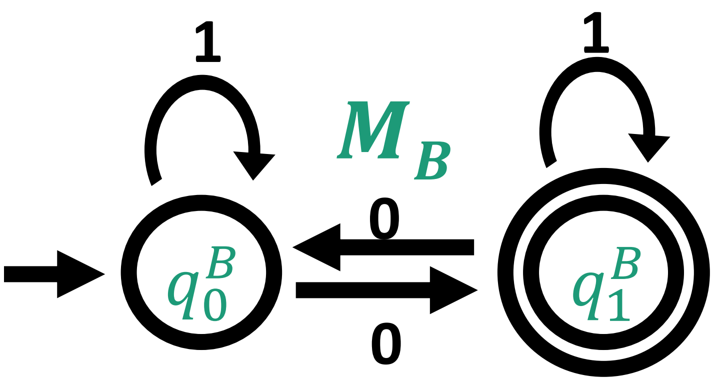
$$\delta(\,(q_A, q_B), \sigma) = (\delta_A(q_A, \sigma), \delta_B(q_B, \sigma))$$

$$q_0 = (q_0^A, q_0^B)$$

$$F = \{(q_A, q_B) \mid q_A \in F_A \text{ or } q_B \in F_B\}$$
$$= F_A \times Q_B \cup Q_A \times F_B$$

Example (cont'd)





M

Intersection

Intersection: $A \cap B = \{ w \mid w \in A \text{ and } w \in B\}$

Theorem: If A and B are regular, then so is $A \cap B$

Proof:

Let $M_A = (Q_A, \Sigma, \delta_A, q_0^A, F_A)$ be a DFA recognizing A and
$M_B = (Q_B, \Sigma, \delta_B, q_0^B, F_B)$ be a DFA recognizing B

Goal: Construct a DFA $M = (Q, \Sigma, \delta, q_0, F)$
that recognizes $A \cap B$

Intersection

Intersection: $A \cap B = \{ w \mid w \in A \text{ and } w \in B\}$

Theorem: If A and B are regular, then so is $A \cap B$

Another Proof:

$A \cap B = \overline{\bar{A} \cup \bar{B}}$

Reverse

Reverse: $A^R = \{w_1 w_2 \cdots w_n | w_n \cdots w_1 \in A\}$

Theorem: If A is regular, then A^R is also regular

Proof idea:

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA recognizing A

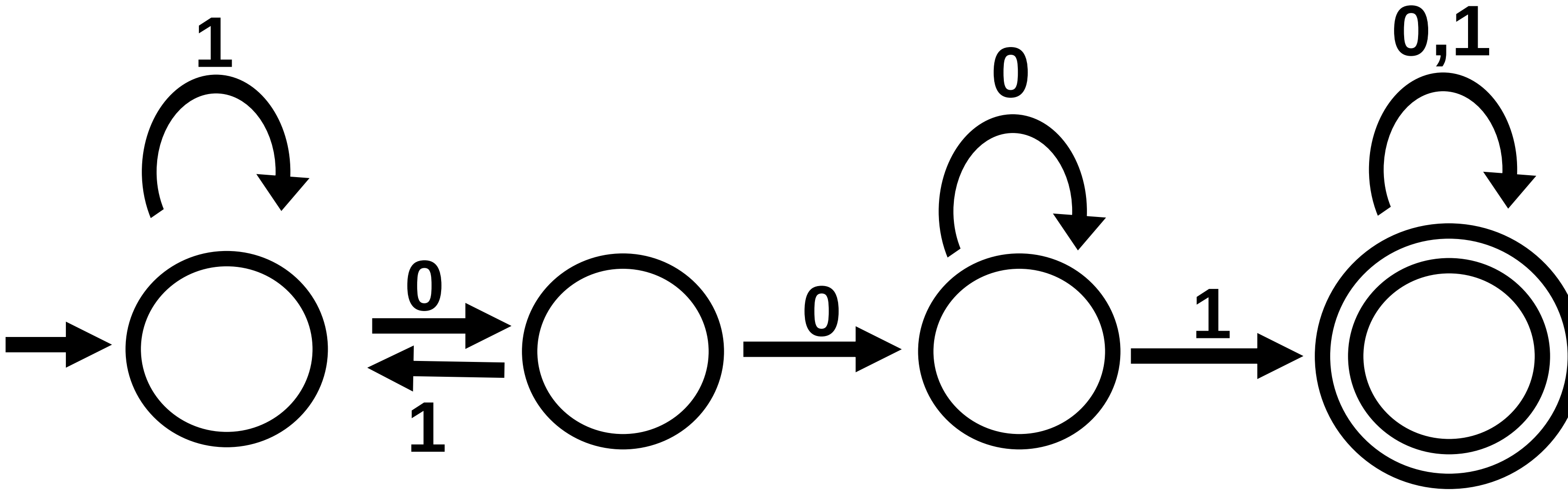
Goal: Construct a DFA $M' = (Q', \Sigma, \delta', q_0', F')$
that recognizes A^R

Define M' as M but

- With the arrows reversed
- With start and accept states swapped

Example (Reverse)

M



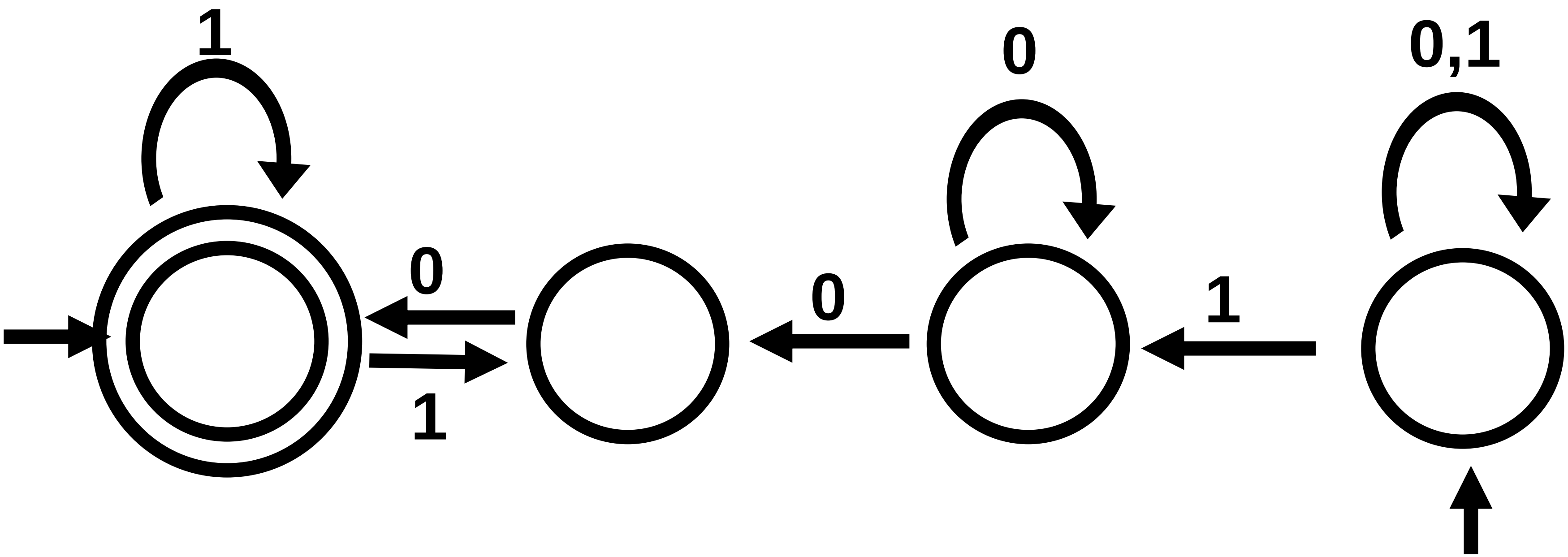
M'

Closure under reverse

M' is not always a DFA!

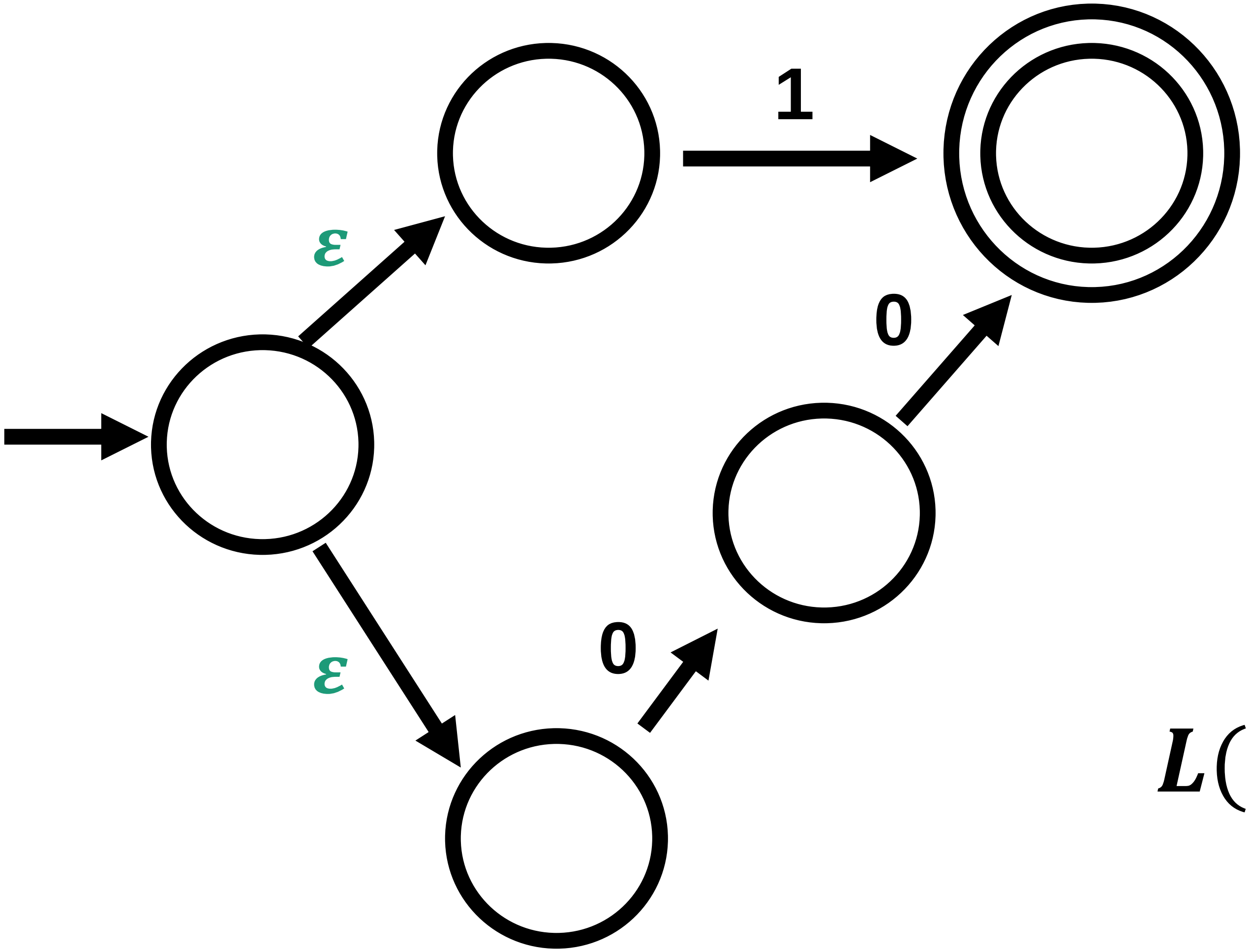
- It might have many start states
- Some states may have too many outgoing edges, or none at all

Nondeterminism



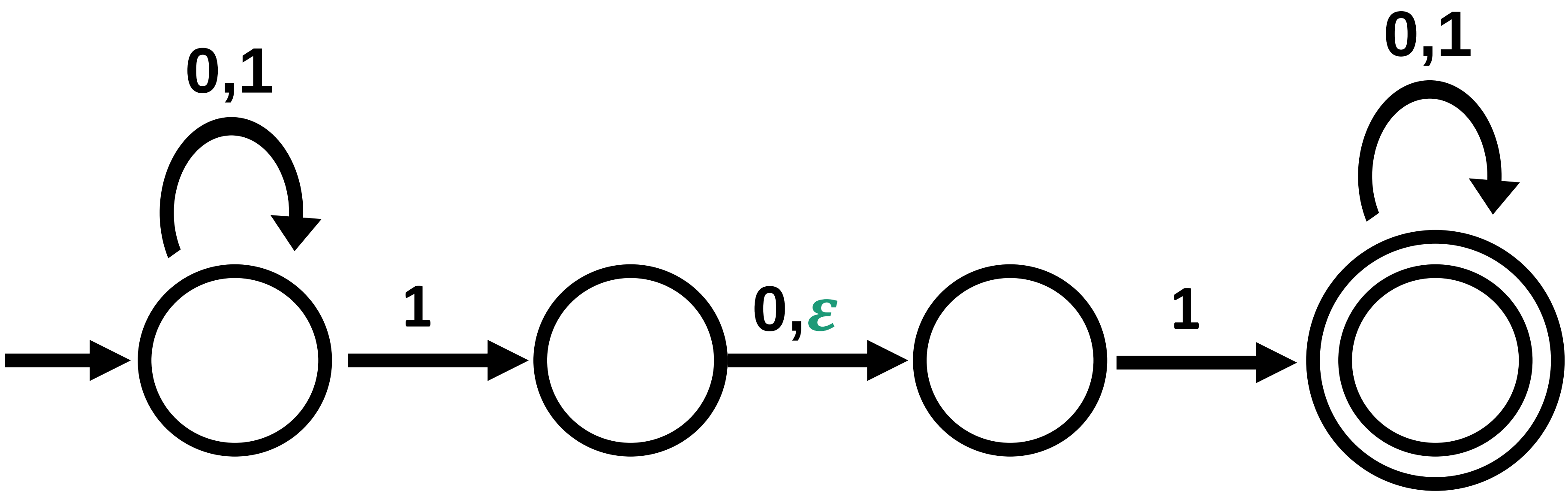
A Nondeterministic Finite Automaton (NFA) accepts if there is a way to make it reach an accept state.

Example



$$L(M) =$$

Example



$L(M) =$

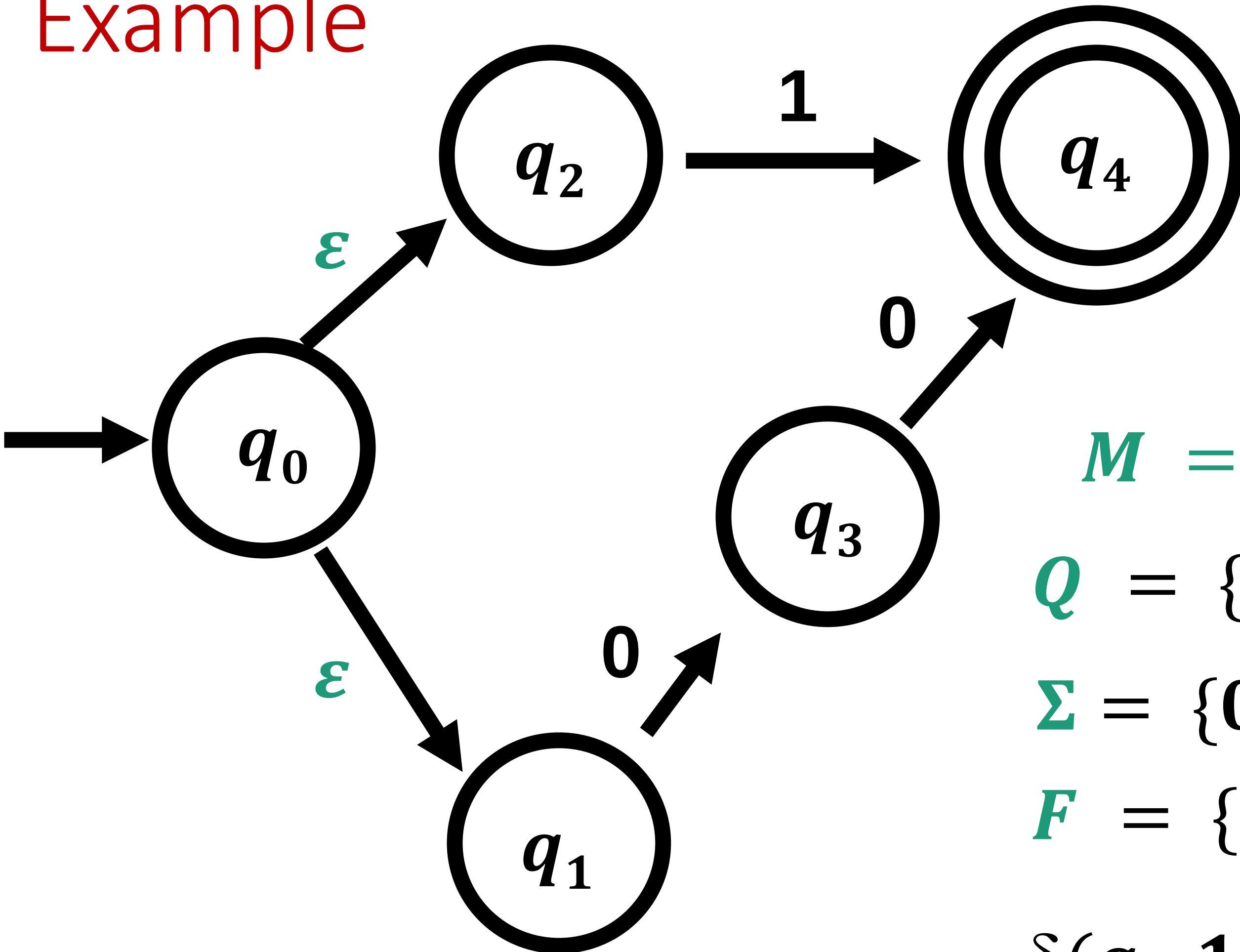
Formal Definition of a NFA

An NFA is a 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$

- Q is the set of states
- Σ is the alphabet
- $\delta : Q \times \Sigma_\varepsilon \to P(Q)$ is the transition function
- $q_0 \in Q$ is the start state
- $F \subseteq Q$ is the set of accept states

M ***accepts*** a string w if **there exists** a path from q_0 to an accept state that can be followed by reading w.

Example



$$M = (Q, \Sigma, \delta, Q_0, F)$$

$$Q = \{q_0, q_1, q_2, q_3, q_4\}$$

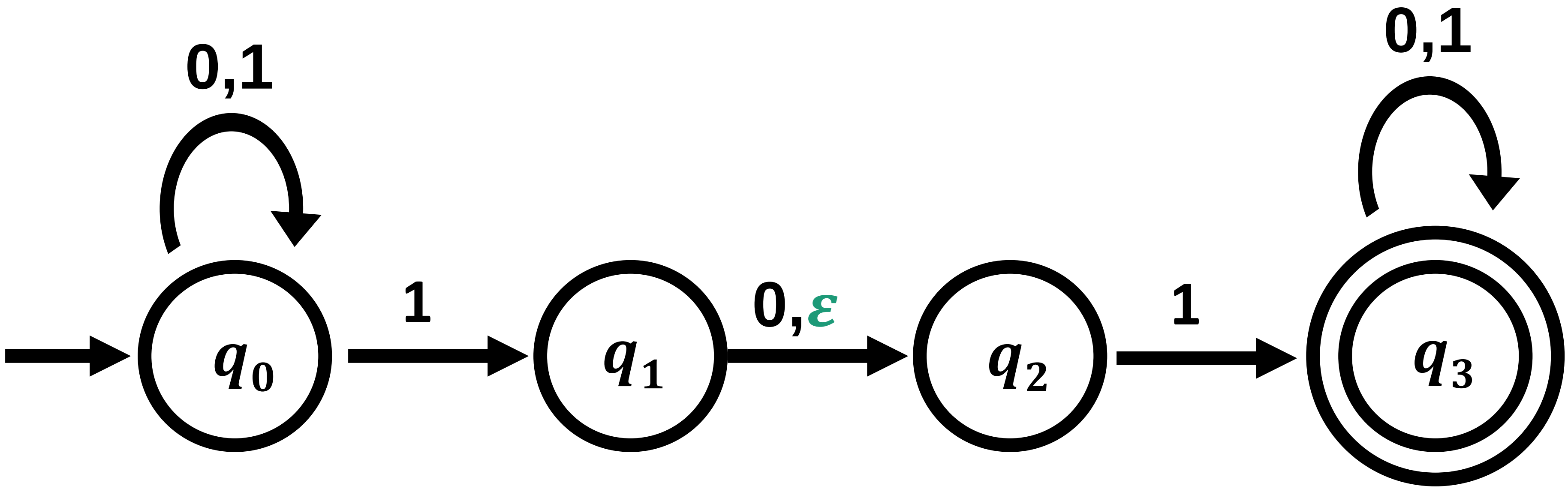
$$\Sigma = \{0, 1\}$$

$$F = \{q_4\}$$

$$\delta(q_2, 1) =$$

$$\delta(q_3, 1) =$$

Example



$N = (Q, \Sigma, \delta, q_0, F)$

$Q = \{q_0, q_1, q_2, q_3\}$

$\Sigma = \{0, 1\}$

$F = \{q_3\}$

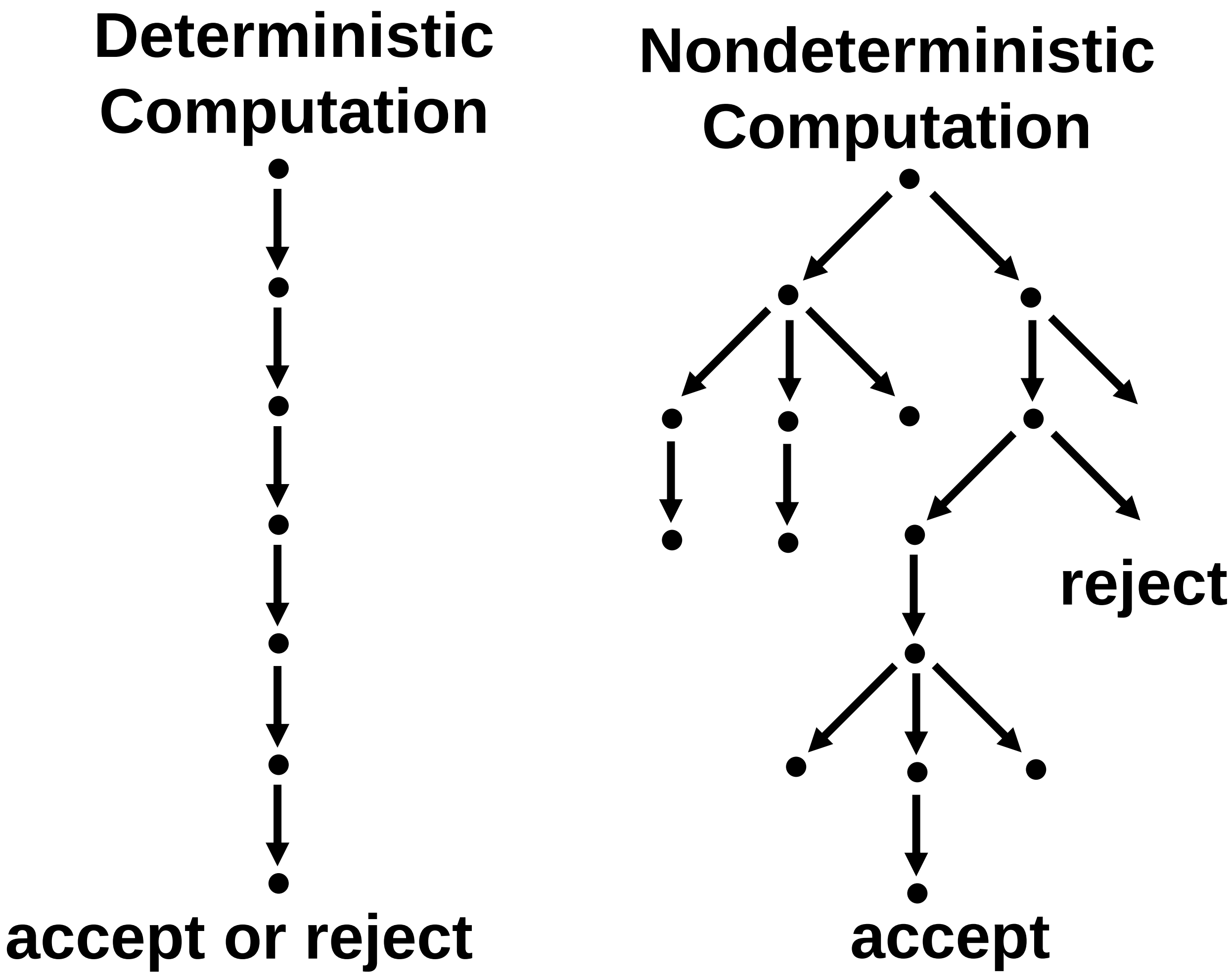
$\delta(q_0, 0) =$

$\delta(q_0, 1) =$

$\delta(q_1, \varepsilon) =$

$\delta(q_2, 0) =$

Nondeterminism



Ways to think about nondeterminism

- **(restricted) parallel computation**
- **tree of possible computations**
- **guessing and verifying the "right" choice**