BU CS 332 – Theory of Computation

Lecture 3:

- Nondeterminism
- Equivalence of NFAs and DFAs
- More on closure properties

Reading:

Sipser Ch 1.1-1.2

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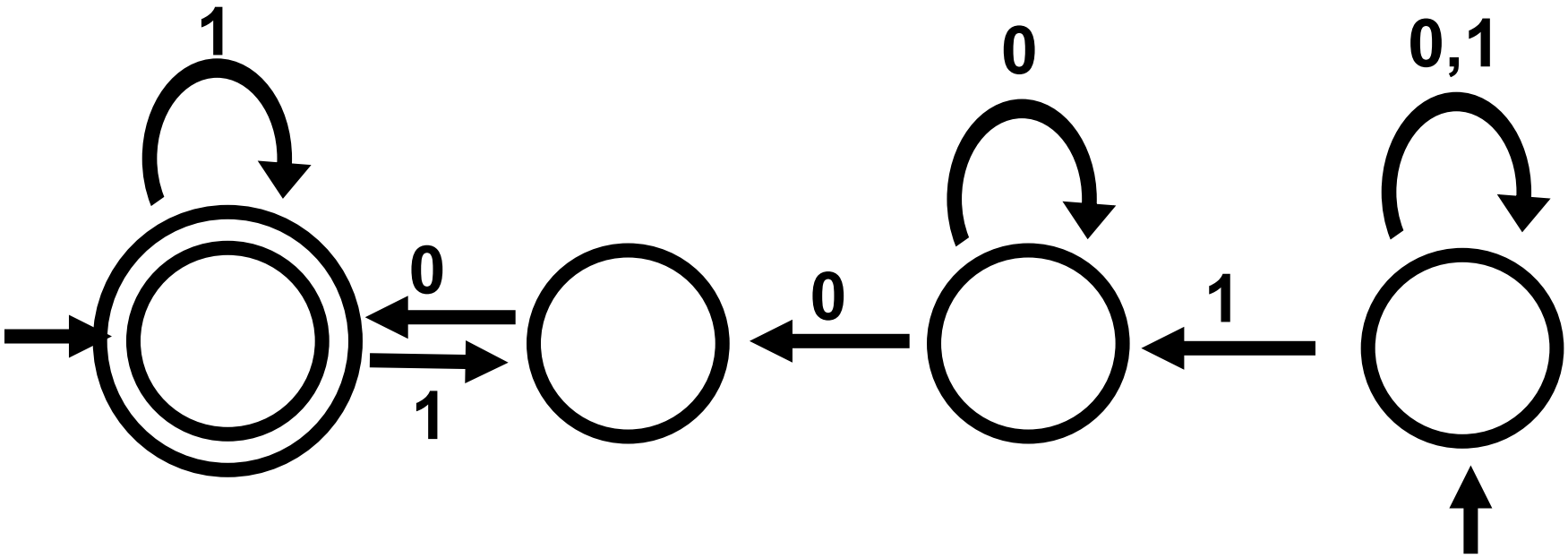
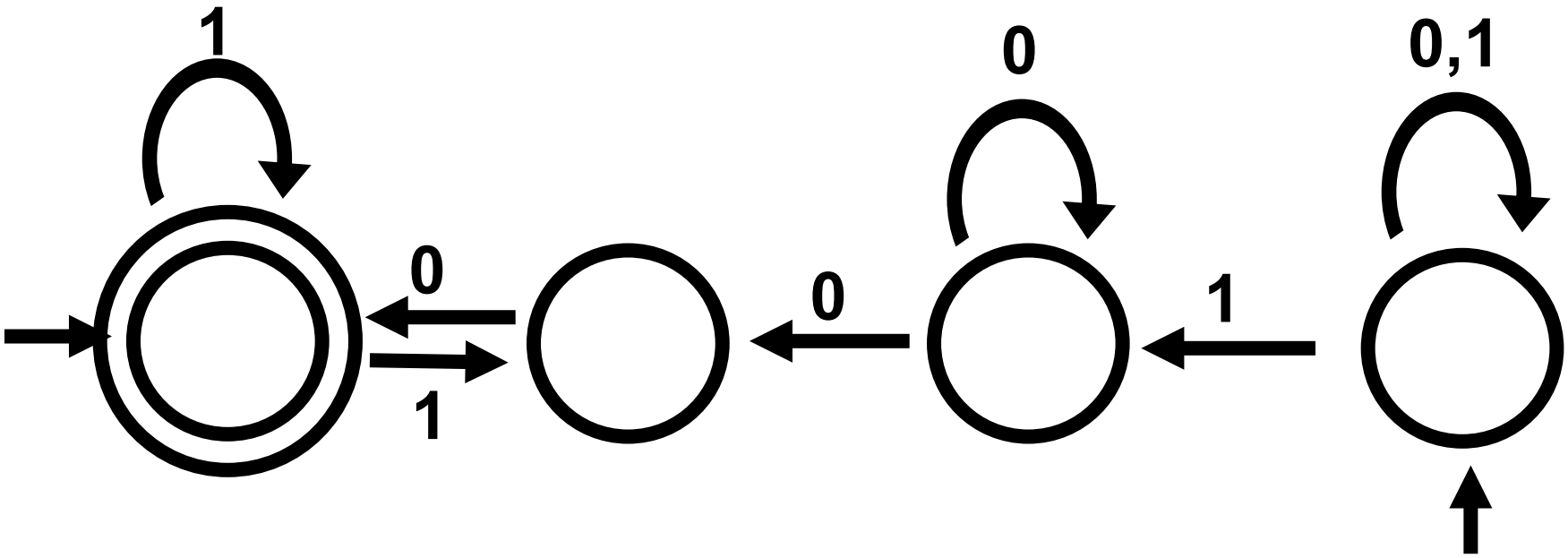
January 29, 2020

Nondeterminism



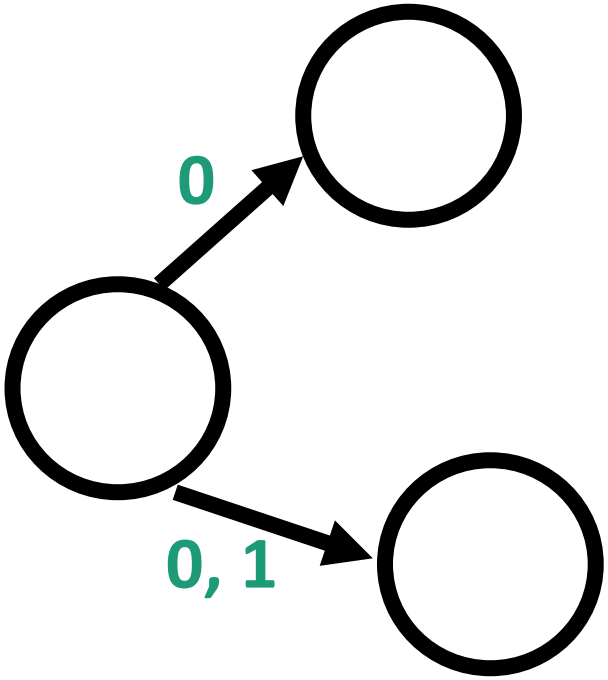
A Nondeterministic Finite Automaton (NFA) accepts if there is a way to make it reach an accept state.

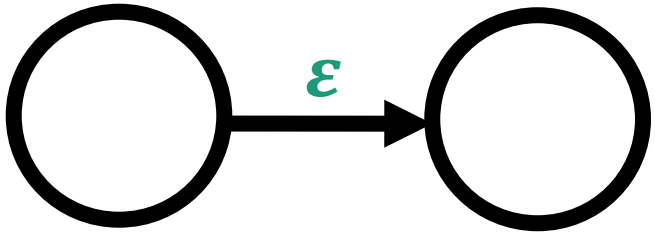
Nondeterminism

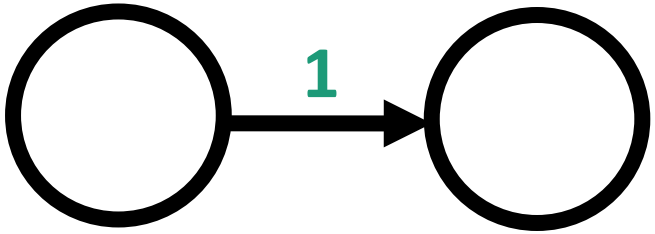


Example: Does this NFA accept the string 1100?

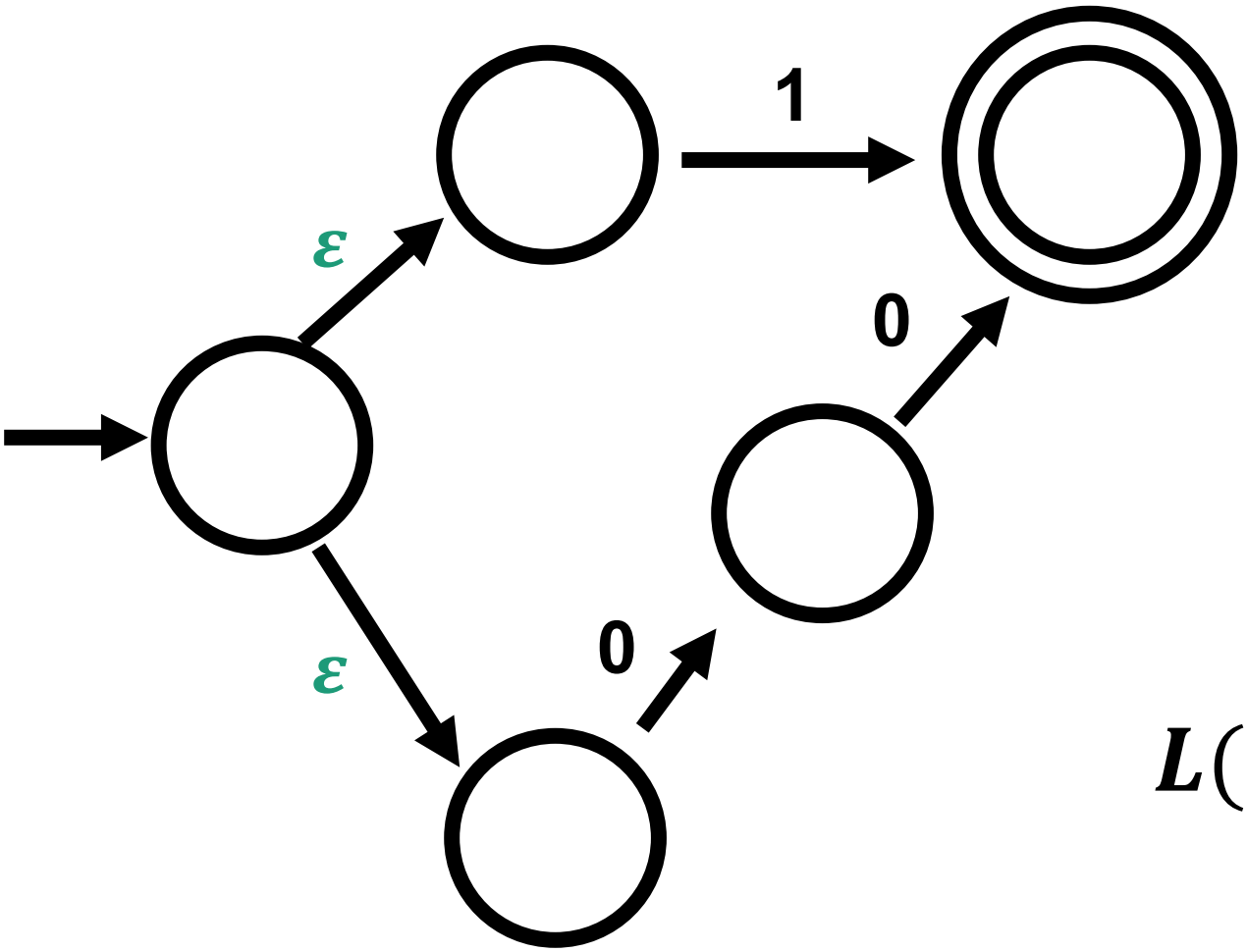
Some special transitions





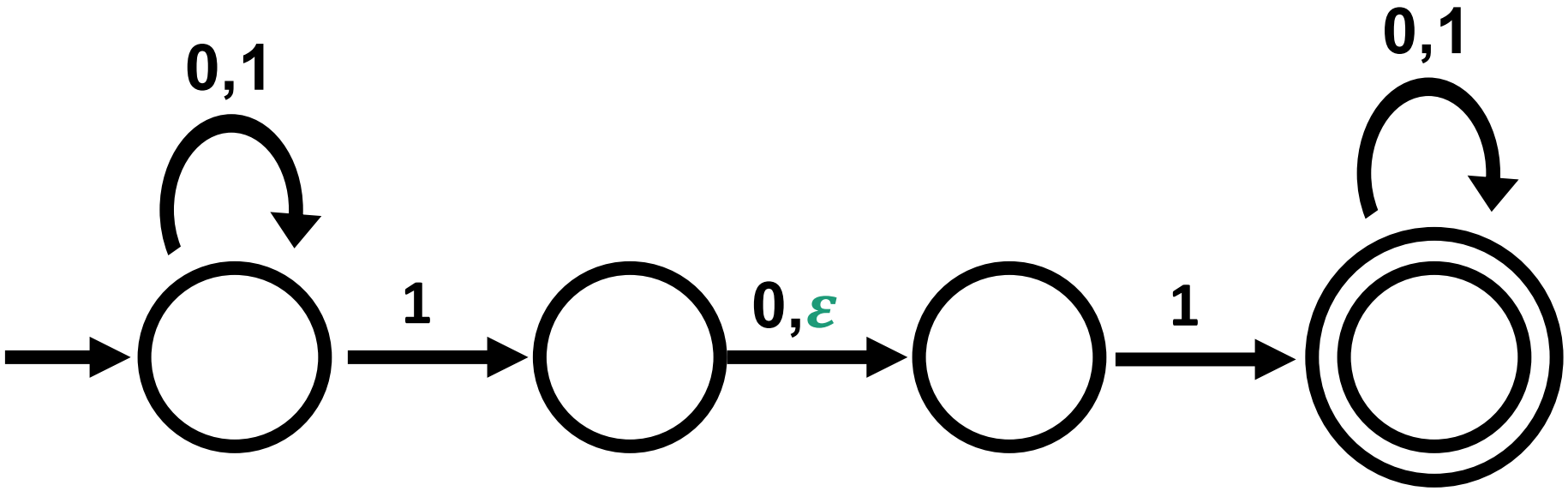


Example



$$L(M) =$$

Example



$$L(M) =$$

Formal Definition of a NFA

An NFA is a 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$

Q is the set of states

Σ is the alphabet

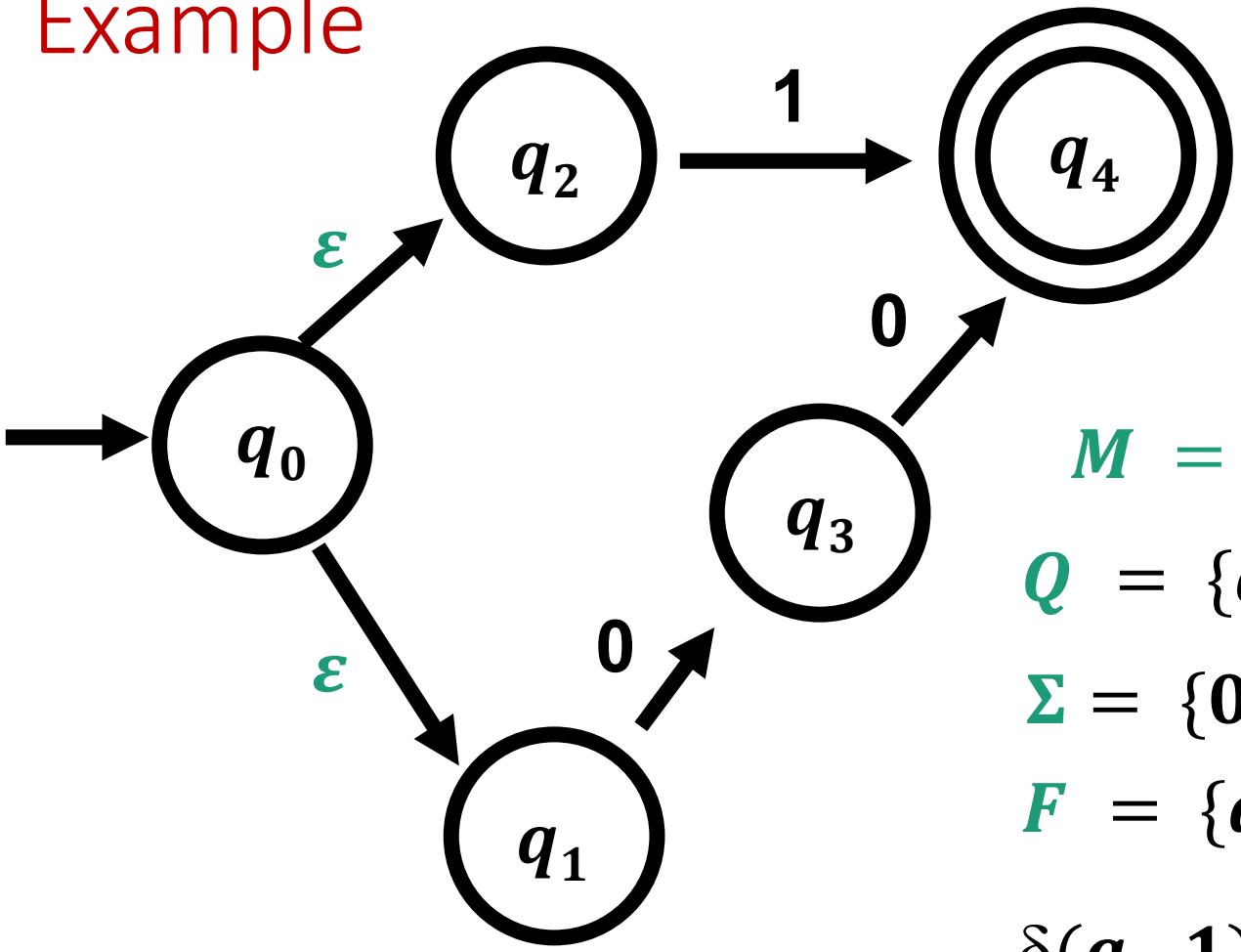
$\delta : Q \times \Sigma_\varepsilon \rightarrow P(Q)$ is the transition function

$q_0 \in Q$ is the start state

$F \subseteq Q$ is the set of accept states

M ***accepts*** a string w if **there exists** a path from q_0 to an accept state that can be followed by reading w.

Example



$$M = (Q, \Sigma, \delta, Q_0, F)$$

$$Q = \{q_0, q_1, q_2, q_3, q_4\}$$

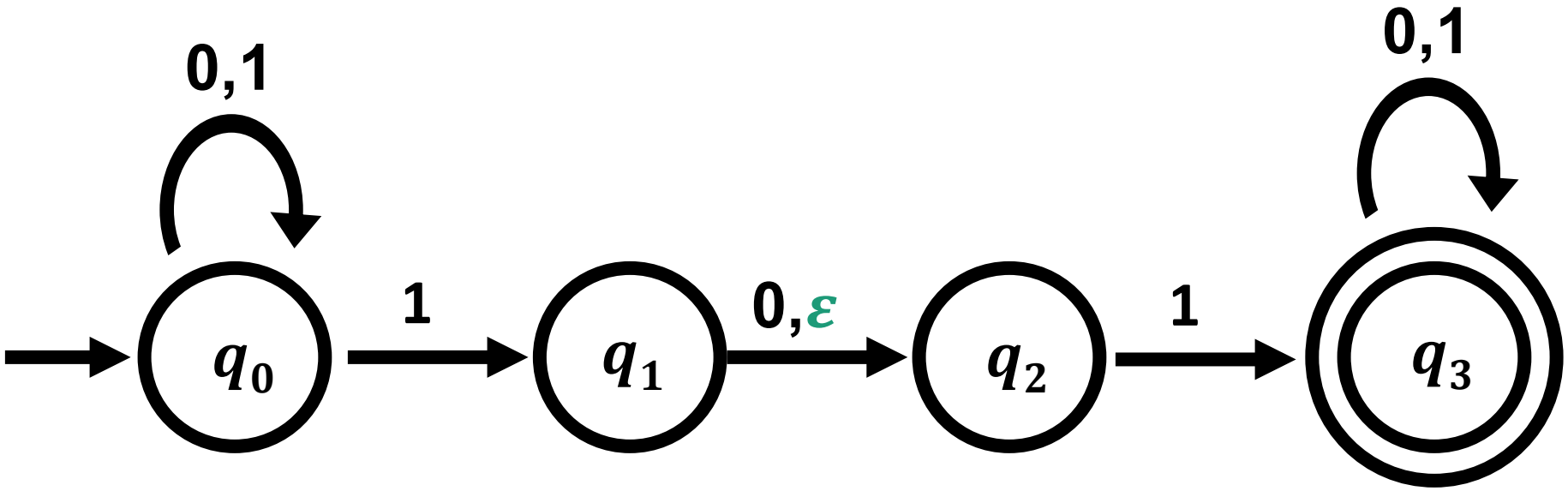
$$\Sigma = \{0, 1\}$$

$$F = \{q_4\}$$

$$\delta(q_2, 1) =$$

$$\delta(q_3, 1) =$$

Example



$N = (Q, \Sigma, \delta, q_0, F)$

$Q = \{q_0, q_1, q_2, q_3\}$

$\Sigma = \{0, 1\}$

$F = \{q_3\}$

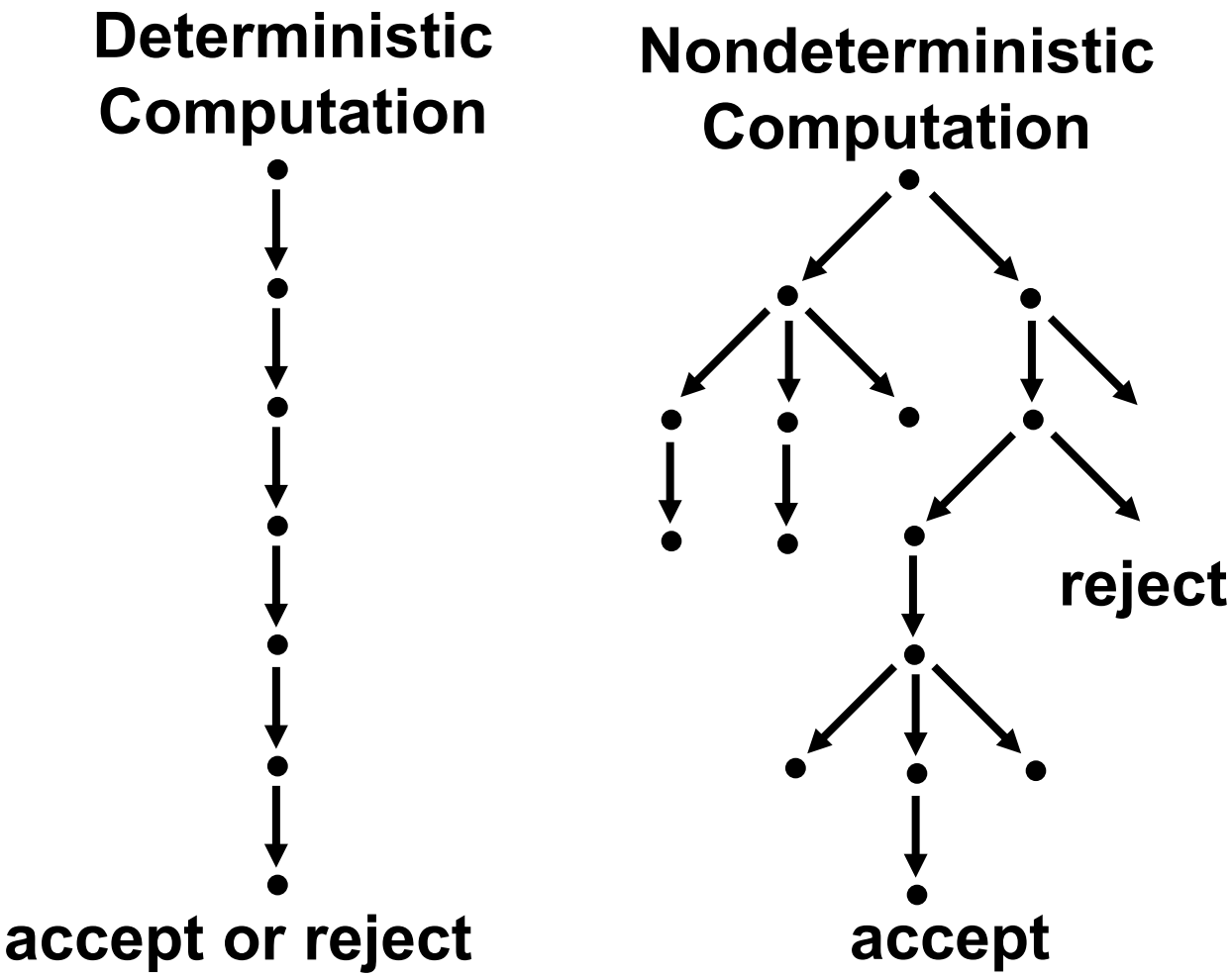
$\delta(q_0, 0) =$

$\delta(q_0, 1) =$?

$\delta(q_1, \varepsilon) =$

$\delta(q_2, 0) =$

Nondeterminism



Ways to think about nondeterminism

- **(restricted) parallel computation**
- **tree of possible computations**
- **guessing and verifying the "right" choice**

Why study NFAs?

- Not really a realistic model of computation: Real computing devices can't really try many possibilities in parallel

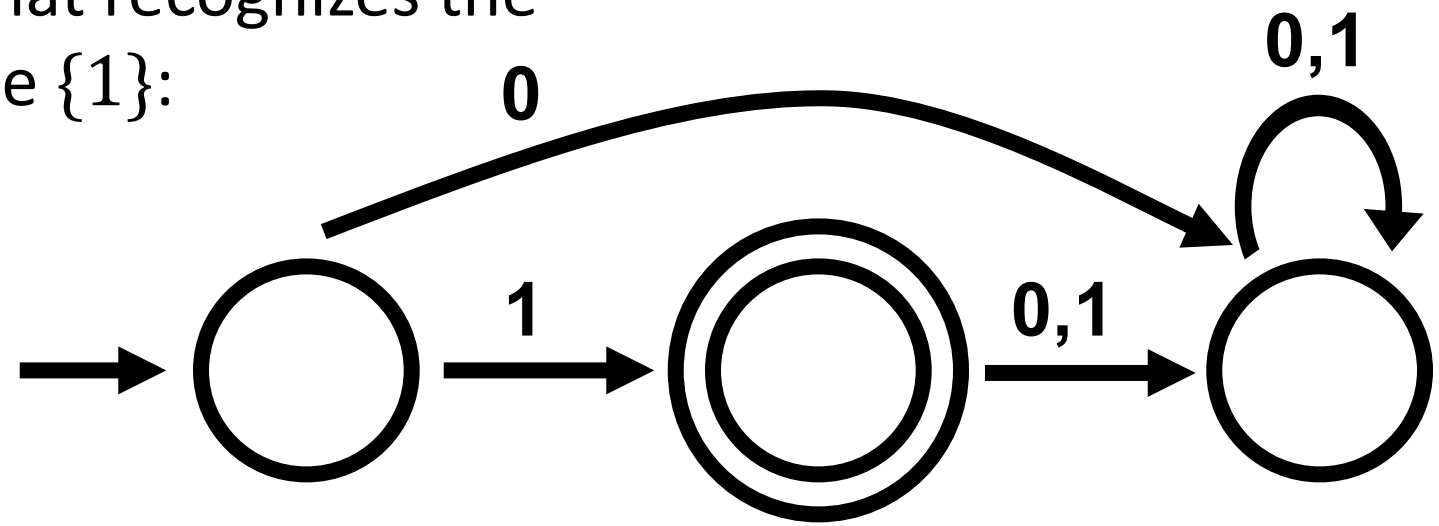
But:

- Useful tool for understanding power of DFAs/regular languages
- NFAs can be simpler than DFAs
- Lets us study "nondeterminism" as a resource

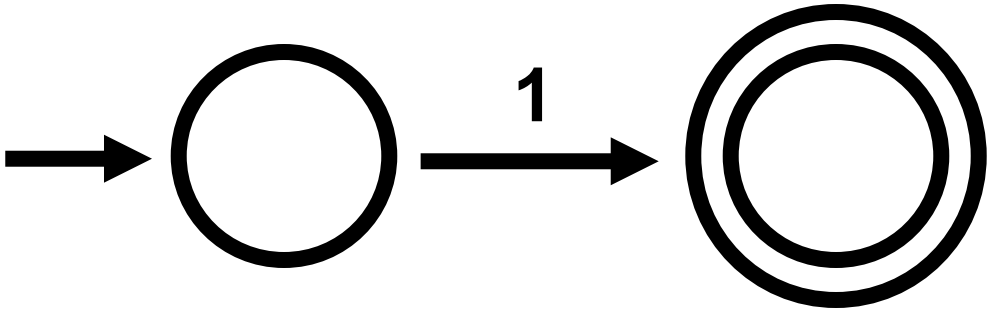
  (cf. P vs. NP)

NFAs can be simpler than DFAs

A DFA that recognizes the language $\{1\}$:



An NFA that recognizes the language $\{1\}$:



Sometimes DFAs **must** be larger

Theorem. Every DFA for the language $\{1\}$ must have at least 3 states.

Proof:

Equivalence of NFAs and DFAs

Equivalence of NFAs and DFAs

Every DFA *is* an NFA, so NFAs are *at least* as powerful as DFAs

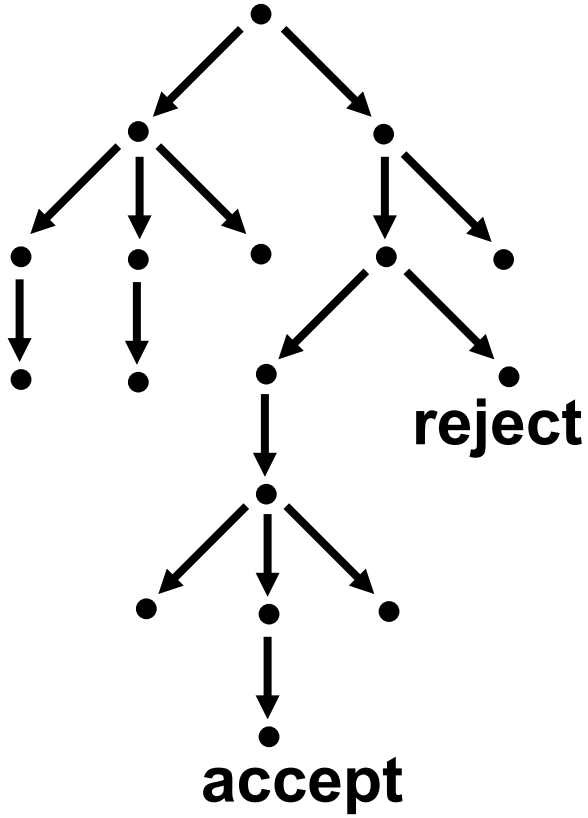
Theorem: For every NFA N, there is a DFA M such that $L(M) = L(N)$

Corollary: A language is regular if and only if it is recognized by an NFA

Equivalence of NFAs and DFAs (Proof)

Let $N = (Q, \Sigma, \delta, q_0, F)$ be an NFA

Goal: Construct DFA $M = (Q', \Sigma, \delta', q_0', F')$ recognizing $L(N)$

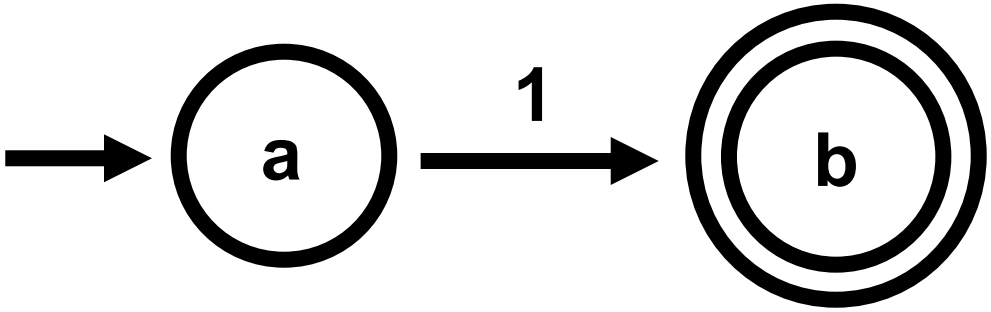


Intuition: Run all threads of N in parallel, maintaining the set of states where all threads are.

Formally: $Q' = P(Q)$

"The Subset Construction"

NFA -> DFA Example



Subset Construction (Formally)

Input: NFA $N = (Q, \Sigma, \delta, q_0, F)$

Output: DFA $M = (Q', \Sigma, \delta', q_0', F')$

Q'

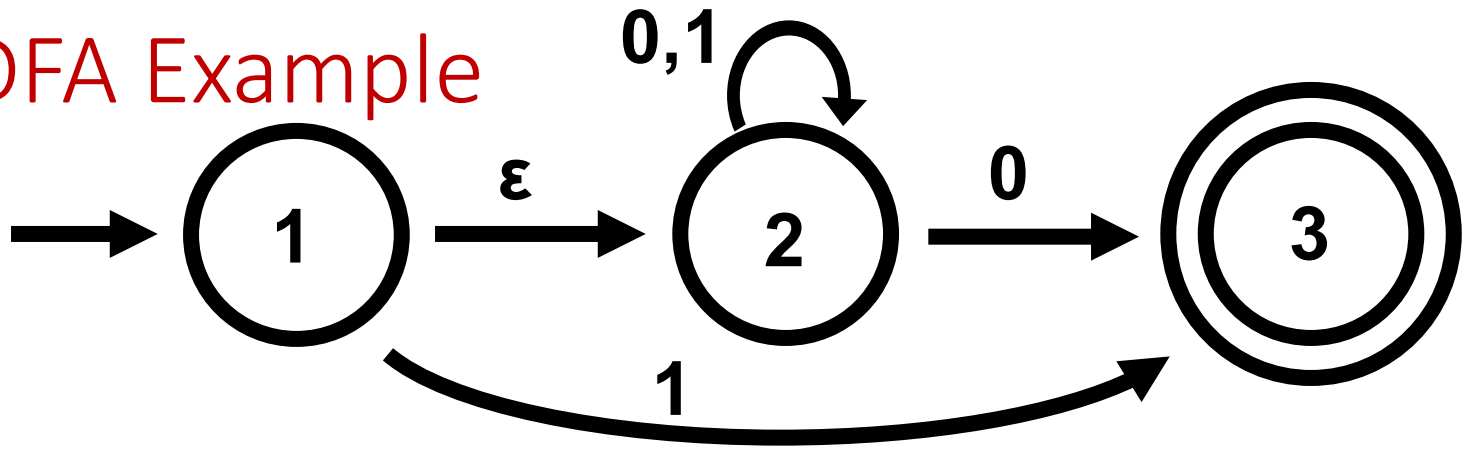
$\delta' : Q' \times \Sigma \rightarrow Q'$

$\delta'(R, \sigma) =$ for all $R \subseteq Q$ and $\sigma \in \Sigma$.

$q_0' =$

$F' =$

NFA -> DFA Example



Subset Construction (Formally)

Input: NFA $N = (Q, \Sigma, \delta, q_0, F)$

Output: DFA $M = (Q', \Sigma, \delta', q_0', F')$

$Q' = P(Q)$

$\delta' : Q' \times \Sigma \rightarrow Q'$

$\delta'(R, \sigma) = \bigcup_{r \in R} \delta(r, \sigma)$ for all $R \subseteq Q$ and $\sigma \in \Sigma$.

$q_0' = \{q_0\}$

$F' = \{ R \in Q' \mid R$ contains some accept state of $N\}$

Proving the Construction Works

Claim: For every string w, running M on w leads to state

$$\{q \in Q | \text{There exists a computation of } N \text{ on input } w \text{ ending at } q\}$$

Proof idea: By induction on $|w|$

Historical Note

Subset Construction introduced in Rabin & Scott's 1959 paper "Finite Automata and their Decision Problems"

1976 ACM Turing Award citation

For their joint paper "Finite Automata and Their Decision Problem," which introduced the idea of nondeterministic machines, which has proved to be an enormously valuable concept. Their (Scott & Rabin) classic paper has been a continuous source of inspiration for subsequent work in this field.

Is this construction the best we can do?

Subset construction converts an n state NFA into a 2^n-state DFA

Could there be a construction that always produces, say, an n^2-state DFA?

Theorem: For every $n \geq 1$, there is a language L_n such that

1. There is an $(n+1)$-state NFA recognizing L_n.
2. There is no DFA recognizing L_n with fewer than 2^n states.

Conclusion: For finite automata, nondeterminism provides an exponential savings over determinism (in the worst case).

More on Closure Properties

Operations on languages

Let $A, B \subseteq \Sigma^*$ be languages. Define

Regular Operations

- Union: $A \cup B$
- Concatenation: $A \circ B = \{ab \mid a \in A, b \in B\}$
- Star: $A^* = \{ a_1 a_2 \ldots a_n \mid n \geq 0 \text{ and } a_i \in A\}$

Complement: $\bar{A}$

Intersection: $A \cap B$

Reverse: $A^R = \{ a_1 a_2 \ldots a_n \mid a_n \ldots a_1 \in A\}$

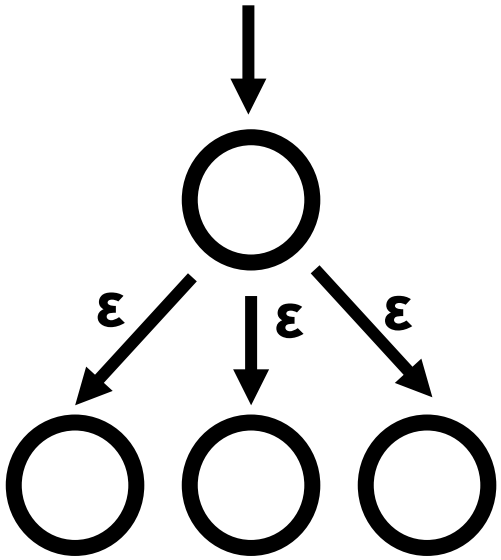
Theorem: The class of regular languages is closed under all six of these operations

Closure under Reverse

Theorem. The reverse of a regular language is also regular

Proof: Let L be a regular language and M be a DFA recognizing it. Construct an NFA M' recognizing L^R:

- Define M' as M with the arrows reversed.
- Make the start state of M be the accept state in M'.
- Make a new start state that goes to all accept states of M by ε-transitions.



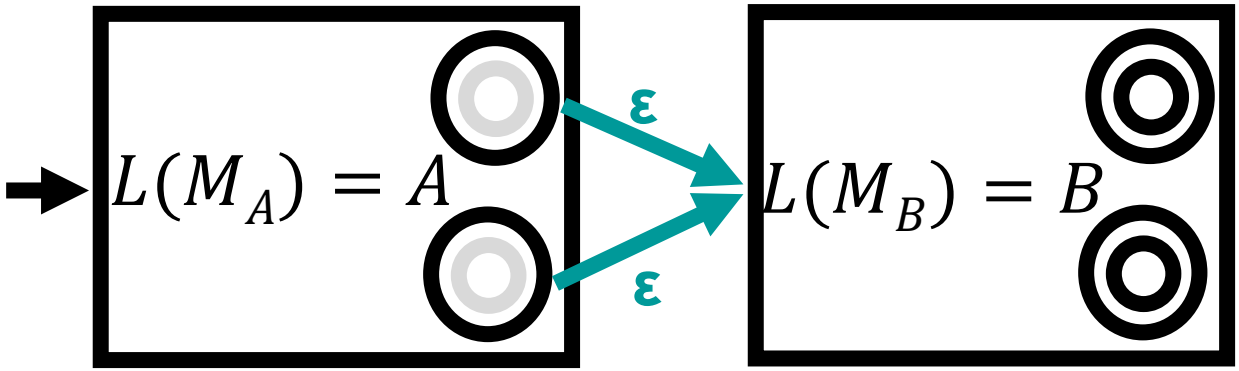
Closure under Concatenation

Concatenation: $A \circ B = \{ ab \mid a \in A \text{ and } b \in B \}$

Theorem. If A and B are regular, $A \circ B$ is also regular.

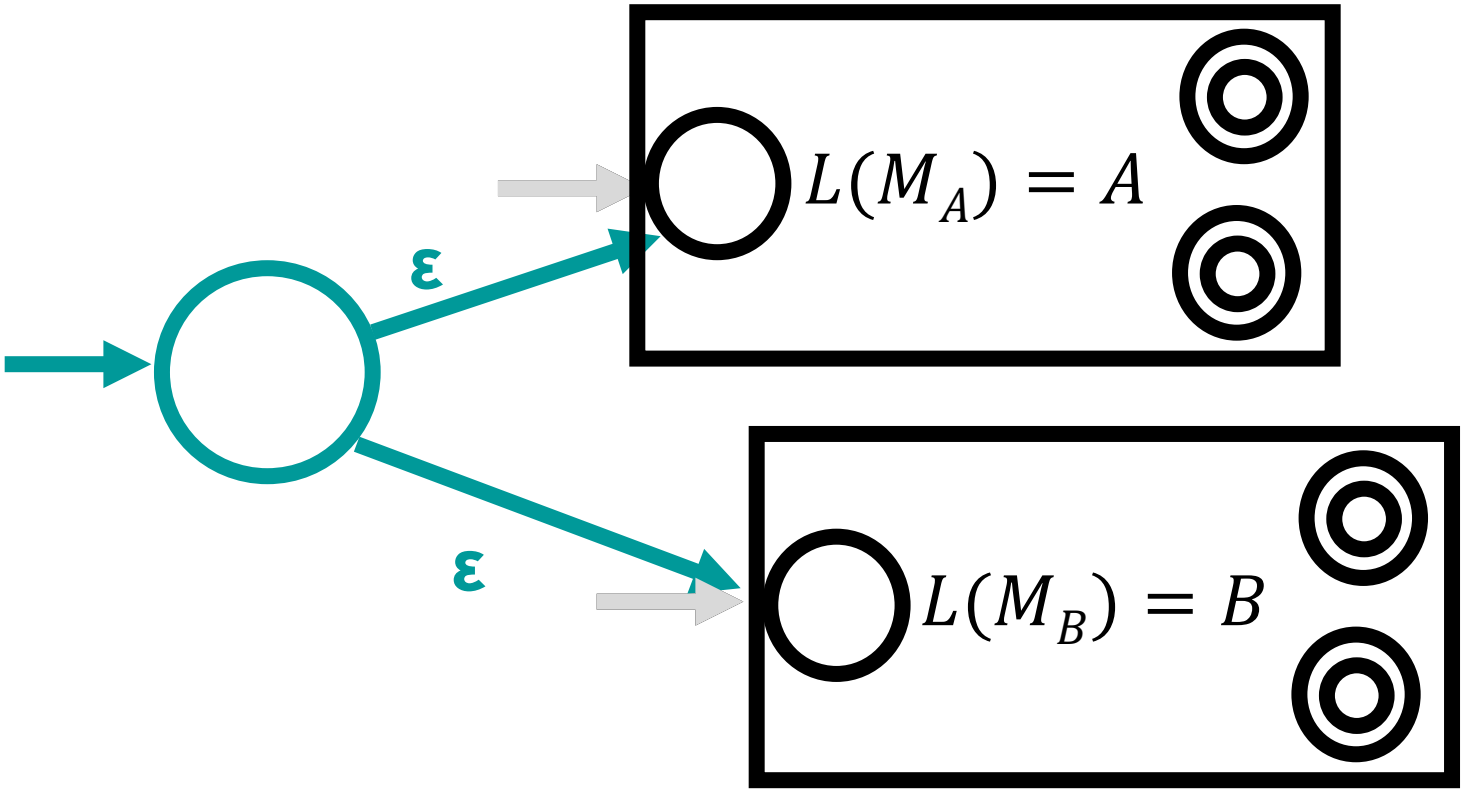
Proof: Given DFAs M_A and M_B, construct NFA by

- Connecting all accept states in M_A to the start state in M_B.
- Make all states in M_A non-accepting.



A Mystery Construction

Given DFAs M_A recognizing A and M_B recognizing B, what does the following NFA recognize?



Closure under Star

Star: $A^* = \{ a_1 a_2 \ldots a_n \mid n \geq 0 \text{ and } a_i \in A \}$

Theorem. If A is regular, A^* is also regular.