Lecture 4:

• Non-regular languages
• Pumping Lemma

Reading:
Sipser Ch 1.4
The Philosophical Question

• We’ve seen techniques for showing that languages are regular

  - Constructing NFAs
  - NFAs
  - Closure properties
  - Next time: Regular expressions

• Could it be the case that every language is regular?
Regular?

Construct an NFA for the following languages

\{0^n1^n \mid 0 < n \leq 2\}

\{0^n1^n \mid 0 < n \leq k\}  \text{ for any } i, j, k \in \mathbb{N}

\{0^n1^n \mid n > 0\}
Proving a language is not regular

Theorem: \( A = \{0^n1^n \mid n > 0\} \) is not regular

Proof: (by contradiction)

Let \( M \) be a DFA with \( k \) states recognizing \( A \)

Consider running \( M \) on input \( 0^k1^k \)
Regular or not?

\[ C = \{ w \mid w \text{ has equal number of 1s and 0s} \} \]

Not regular

\[ D = \{ w \mid w \text{ has equal number of 10s and 01s} \} \]

Regular!
The Pumping Lemma

A systematic way to prove that a language is not regular
Why do we teach this?

Cons:

• The statement is difficult (5 quantifiers!)
• Some non-regular languages can still be pumped

Pros:

• Proof illuminates essential structure of finite automata
• Generalizes to other models of computation / classes of languages (CFLs, self-assembly)
• Applying it can be fun!
Intuition for the Pumping Lemma

Imagine a **DFA** with $p$ states that recognizes strings of length $> p$

Idea: If you can go around the cycle once, you can go around 0 or 2,3,4... times
Pumping Lemma (Informal)

Let $L$ be a regular language. Let $w$ be a “long enough” string in $L$.

Then we can write $w = xyz$ such that $xy^iz \in L$ for every $i \geq 0$.

$i = 0$: $\varepsilon$

$i = 1$: $xy\varepsilon$

$i = 2$: $xxy\varepsilon$

$i = 3$: $xxyy\varepsilon$
Pumping Lemma (Formal)

Let $L$ be a regular language.

Then there exists a “pumping length” $p$ such that

For every $w \in L$ where $|w| \geq p$,

$w$ can be split into three parts $w = xyz$ where:

1. $|y| > 0 \in 
2. |xy| \leq p
3. xy^iz \in L$ for all $i \geq 0$

Example:
Let $L = \{w \mid$ all $a$’s in $w$ appear before all $b$’s$\}; p = 1$

$w = aabb$

$x = \epsilon$

$y = a$

$z = ab$

$w = aabb$

$x = \epsilon$

$y = a$

$z = aabb$
Using the Pumping Lemma

**Theorem:** $A = \{0^n1^n \mid n > 0\}$ is not regular

**Proof:** (by contradiction)

Assume instead that $A$ is regular. Then $A$ has a pumping length $p$.

What happens if we try to pump $0^p1^p$?

If $A$ is regular, $w$ can be split into $w = xyz$, where

1. $|y| > 0 \Rightarrow y = 0^i \text{ for } 0 < i \leq p$
2. $|xy| \leq p \Rightarrow xy = 0^k \text{ for some } 0 \leq k \leq p$
3. $xy^iz \in A$ for all $i \geq 0$
General Strategy for proving $L$ is not regular

Proof by contradiction: assume $L$ is regular.
Then there is a pumping length $p$.

1) Choose a string $w$, $|w| > p$

2) Show that for every decomposition $w = x y z$ (satisfying conditions of PL), $w$ cannot be pumped

3) $x = \Rightarrow L$ non-regular
Pumping Lemma as a game

1. **YOU** pick the language $L$ to be proved nonregular.
2. **ADVERSARY** picks a possible pumping length $p$.
3. **YOU** pick $w$ of length at least $p$.
4. **ADVERSARY** divides $w$ into $x, y, z$, obeying rules of the Pumping Lemma: $|y| > 0$ and $|xy| \leq p$.
5. **YOU** win by finding $i \geq 0$, for which $xy^iz$ is not in $L$.

If *regardless* of how the **ADVERSARY** plays this game, you can always win, then $L$ is nonregular.
Example: Palindromes

Claim: $L = \{ww^R \mid w \in \{0,1\}^*\}$ is not regular

Proof: Assume $L$ is regular with pumping length $p$

1. Find $w \in L$ with $|w| > p$
   \[ w = 0^p 1^p 1^p 0^p \]

2. Show that $w$ cannot be pumped

   Intuitively
Example: Palindromes

Claim: $L = \{ww^R \mid w \in \{0,1\}^*\}$ is not regular

Proof: Assume $L$ is regular with pumping length $p$

1. Find $w \in L$ with $|w| > p$

2. Show that $w$ cannot be pumped
   Formally  If $w = xyz$ with $|xy| \leq p$, then...
Now you try!

Claim: $L = \{0^i1^j \mid i > j \geq 0\}$ is not regular

Proof: Assume $L$ is regular with pumping length $p$

1. Find $w \in L$ with $|w| > p$

2. Show that $w$ cannot be pumped

Intuitively