BU CS 332 – Theory of Computation

Lecture 4:

• Non-regular languages
• Pumping Lemma

Reading:
Sipser Ch 1.4

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The Philosophical Question

• We’ve seen techniques for showing that languages are regular

• Could it be the case that every language is regular?
Regular?

Construct an NFA for the following languages

\( \{0^n1^n \mid 0 < n \leq 2\} \)

\( \{0^n1^n \mid 0 < n \leq k\} \)

\( \{0^n1^n \mid n > 0\} \)
Proving a language is not regular

**Theorem:** $A = \{0^n1^n \mid n > 0\}$ is not regular

**Proof:** (by contradiction)

Let $M$ be a DFA with $k$ states recognizing $A$

Consider running $M$ on input $0^k1^k$
Regular or not?

\[ C = \{ w \mid w \text{ has equal number of 1s and 0s} \} \]

\[ D = \{ w \mid w \text{ has equal number of 10s and 01s} \} \]
The Pumping Lemma

A **systematic** way to prove that a language is not regular
Why do we teach this?

Cons:
• The statement is difficult (5 quantifiers!)
• Some non-regular languages can still be pumped

Pros:
• Proof illuminates essential structure of finite automata
• Generalizes to other models of computation / classes of languages (CFLs, self-assembly)
• Applying it can be fun!
Intuition for the Pumping Lemma

Imagine a DFA with $p$ states that recognizes strings of length $> p$

**Idea:** If you can go around the cycle once, you can go around 0 or 2,3,4... times
Pumping Lemma (Informal)

Let $L$ be a regular language. Let $w$ be a “long enough” string in $L$.

Then we can write $w = xyz$ such that $xy^iz \in L$ for every $i \geq 0$.

$i = 0:$

$i = 1:$

$i = 2:$

$i = 3:$
Pumping Lemma (Formal)

Let $L$ be a regular language.

Then there exists a “pumping length” $p$ such that

For every $w \in L$ where $|w| \geq p$,

- $w$ can be split into three parts $w = xyz$ where:

1. $|y| > 0$
2. $|xy| \leq p$
3. $xy^iz \in L$ for all $i \geq 0$

Example:

Let $L = \{w \mid$ all $a$’s in $w$ appear before all $b$’s$\}$; $p = 1$
Using the Pumping Lemma

**Theorem:** $A = \{0^n1^n \mid n > 0\}$ is not regular

**Proof:** (by contradiction)

Assume instead that $A$ is regular. Then $A$ has a pumping length $p$.

What happens if we try to pump $0^p1^p$?

If $A$ is regular, $w$ can be split into $w = xyz$, where

1. $|y| > 0$
2. $|xy| \leq p$
3. $xy^iz \in A$ for all $i \geq 0$
General Strategy for proving $L$ is not regular

Proof by contradiction: assume $L$ is regular. Then there is a pumping length $p$. 
Pumping Lemma as a game

1. **YOU** pick the language $L$ to be proved nonregular.
2. **ADVERSARY** picks a possible pumping length $p$.
3. **YOU** pick $w$ of length at least $p$.
4. **ADVERSARY** divides $w$ into $x, y, z$, obeying rules of the Pumping Lemma: $|y| > 0$ and $|xy| \leq p$.
5. **YOU** win by finding $i \geq 0$, for which $xy^iz$ is not in $L$.

If *regardless* of how the **ADVERSARY** plays this game, you can always win, then $L$ is nonregular.
Example: Palindromes

Claim: $L = \{ww^R \mid w \in \{0,1\}^*\}$ is not regular

Proof: Assume $L$ is regular with pumping length $p$

1. Find $w \in L$ with $|w| > p$

2. Show that $w$ cannot be pumped

Intuitively
Example: Palindromes

Claim: \( L = \{ww^R \mid w \in \{0,1\}^*\} \) is not regular

Proof: Assume \( L \) is regular with pumping length \( p \)

1. Find \( w \in L \) with \( |w| > p \)

2. Show that \( w \) cannot be pumped
   
   Formally  If \( w = xyz \) with \( |xy| \leq p \), then...
Claim: $L = \{0^i1^j \mid i > j \geq 0\}$ is not regular

Proof: Assume $L$ is regular with pumping length $p$

1. Find $w \in L$ with $|w| > p$

2. Show that $w$ cannot be pumped

Intuitively
Claim: $L = \{0^i 1^j \mid i > j \geq 0\}$ is not regular

Proof: Assume $L$ is regular with pumping length $p$

1. Find $w \in L$ with $|w| > p$

2. Show that $w$ cannot be pumped
   Formally If $w = xyz$ with $|xy| \leq p$, then...
Claim: \( BALANCED = \{w \mid w \text{ has an equal # of 0s and 1s}\} \) is not regular

Proof: Assume \( L \) is regular with pumping length \( p \)

1. Find \( w \in L \) with \( |w| > p \)

2. Show that \( w \) cannot be pumped
   Formally  If \( w = xyz \) with \( |xy| \leq p \), then...
Reusing a Proof

Pumping a language can be lots of work...
Let’s try to reuse that work!

How else might we show that $BALANCED$ is regular?

$\{0^n1^n \mid n \geq 0\} = BALANCED \cap \{w \mid \text{all 0s in } w \text{ appear before all 1s}\}$
Using Closure Properties

If $A$ is not regular, we can show a related language $B$ is not regular

By contradiction: If $B$ is regular, then $B \cap C (= A)$ is regular.

But $A$ is not regular so neither is $B$!
Example

Prove $B = \{0^i1^j | i \neq j\}$ is not regular using nonregular language $A = \{0^n1^n | n \geq 0\}$ and regular language $C = \{w \mid \text{all 0s in } w \text{ appear before all 1s}\}$