BU CS 332 – Theory of Computation

Lecture 4:

- Non-regular languages
- Pumping Lemma

Reading:

Sipser Ch 1.4

Mark Bun

February 3, 2020

The Philosophical Question

- We’ve seen techniques for showing that languages are regular

- Could it be the case that every language is regular?

Regular?

Construct an NFA for the following languages

$\{0^n 1^n \mid 0 < n \leq 2\}$

$\{0^n 1^n \mid 0 < n \leq k\}$

$\{0^n 1^n \mid n > 0\}$

Proving a language is not regular

Theorem: $A = \{0^n 1^n \mid n > 0\}$ is not regular

Proof: (by contradiction)

Let M be a DFA with k states recognizing A

Consider running M on input $0^k 1^k$

Regular or not?

$C \;=\; \{\, w \mid w$ has equal number of 1s and 0s$\}$

$D \;=\; \{\, w \mid w$ has equal number of 10s and 01s$\}$

The Pumping Lemma

A **systematic** way to prove that a language is not regular

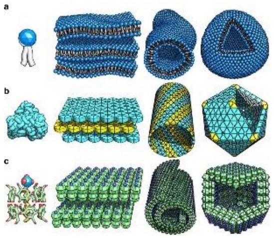
Why do we teach this?

Cons:

- The statement is difficult (5 quantifiers!)
- Some non-regular languages can still be pumped

Pros:

- Proof illuminates essential structure of finite automata
- Generalizes to other models of computation / classes of languages (CFLs, self-assembly)
- Applying it can be fun!



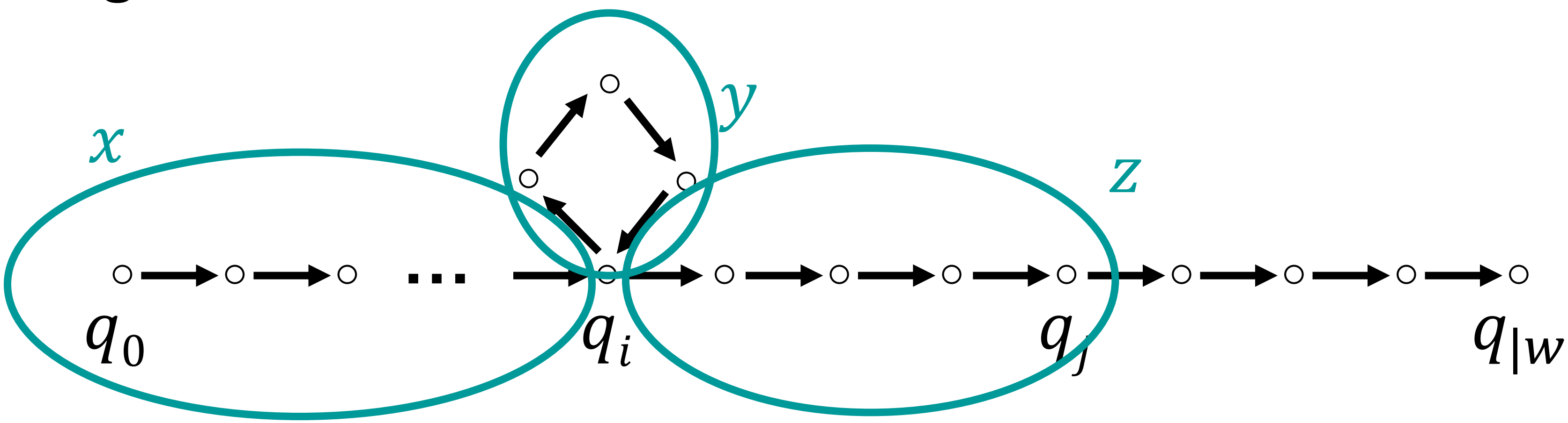
Intuition for the Pumping Lemma

Imagine a **DFA** with p states that recognizes strings of length $> \ p$

Idea: If you can go around the cycle once, you can go around 0 or 2,3,4… times

Pumping Lemma (Informal)

Let L be a regular language. Let w be a "long enough" string in L.



Then we can write $w = xyz$ such that $xy^iz \in L$ for every $i \geq 0$.

$i = 0$:

$i = 1$:

$i = 2$:

$i = 3$:

Pumping Lemma (Formal)

Let L be a regular language.

Then there exists a "pumping length" p such that

For every $w \in L$ where $|w| \geq p$,

w can be split into three parts $w = xyz$ where:

1. $|y| > 0$
2. $|xy| \leq p$
3. $xy^iz \in L$ for all $i \geq 0$

Example:
Let $L = \{w \mid$ all a's in w appear before all b's$\}$; $p = 1$

Using the Pumping Lemma

Theorem: $A = \{0^n 1^n \mid n > 0\}$ is not regular

Proof: (by contradiction)

Assume instead that A is regular. Then A has a pumping length p.

What happens if we try to pump $0^p 1^p$?

If A is regular, w can be split into $w = xyz$, where

1. $|y| > 0$
2. $|xy| \leq p$
3. $xy^iz \in A$ for all $i \geq 0$

General Strategy for proving L is not regular

Proof by contradiction: assume L is regular.

Then there is a pumping length p.

Pumping Lemma as a game

1. YOU pick the language L to be proved nonregular.
2. ADVERSARY picks a possible pumping length p.
3. YOU pick w of length at least p.
4. ADVERSARY divides w into x, y, z, obeying rules of the Pumping Lemma: $|y| > 0$ and $|xy| \leq p$.
5. YOU win by finding $i \geq 0$, for which $xy^i z$ is not in L.

If *regardless* of how the ADVERSARY plays this game, you can always win, then L is nonregular

Example: Palindromes

<u>Claim:</u> $L = \{ww^R \mid w \in \{0,1\}^*\}$ is not regular

<u>Proof:</u> Assume L is regular with pumping length p

1. Find $w \in L$ with $|w| > p$

2. Show that w cannot be pumped
   <u>Intuitively</u>

Example: Palindromes

Claim: $L = \{ww^R \mid w \in \{0,1\}^*\}$ is not regular

Proof: Assume L is regular with pumping length p

1. Find $w \in L$ with $|w| > p$

2. Show that w cannot be pumped
   Formally If $w = xyz$ with $|xy| \leq p$, then…

Now you try!

Claim: $L = \{0^i 1^j \mid i > j \geq 0\}$ is not regular

Proof: Assume L is regular with pumping length p

1. Find $w \in L$ with $|w| > p$

2. Show that w cannot be pumped
   Intuitively

Now you try!

Claim: $L = \{0^i 1^j \mid i > j \geq 0\}$ is not regular

Proof: Assume L is regular with pumping length p

1. Find $w \in L$ with $|w| > p$

2. Show that w cannot be pumped
   Formally   If $w = xyz$ with $|xy| \leq p$, then...

Choosing wisely

Claim: $BALANCED = \{w \mid w \text{ has an equal \# of 0s and 1s}\}$ is not regular

Proof: Assume L is regular with pumping length p

1. Find $w \in L$ with $|w| > p$

2. Show that w cannot be pumped
   Formally If $w = xyz$ with $|xy| \leq p$, then...

Reusing a Proof

Pumping a language can be lots of work…

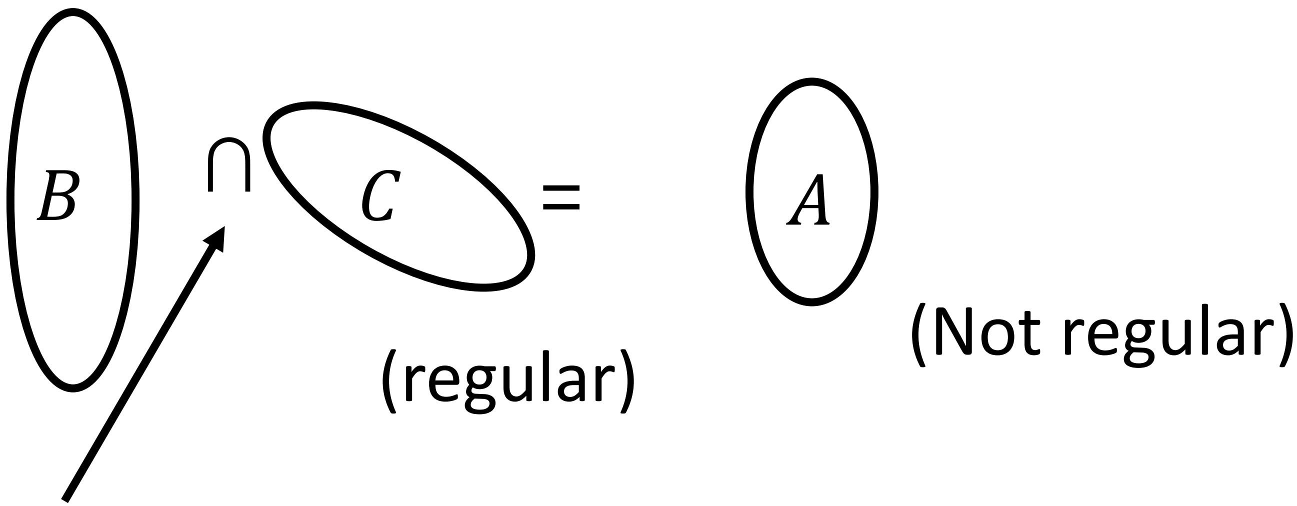
Let’s try to reuse that work!

How else might we show that $BALANCED$ is regular?

$\{0^n 1^n \mid n \geq 0\}$ = $BALANCED \cap \{w \mid$ all 0s in w appear before all 1s$\}$

Using Closure Properties

If A is not regular, we can show a related language B is not regular



any of $\{\circ, \cup, \cap\}$ or, for one language, $\{\neg, {}^{R}, *\}$

<u>By contradiction:</u> If B is regular, then $B \cap C$ $(= A)$ is regular.

But A is not regular so neither is B!

Example

Prove $B = \{0^i 1^j | i \neq j\}$ is not regular
using nonregular language $A = \{0^n 1^n | n \geq 0\}$
and regular language
$C = \{w \mid$ all 0s in w appear before all 1s$\}$