BU CS 332 – Theory of Computation

Lecture 5:

- More on pumping
- Regular expressions
- Regular expressions = regular languages

Reading:

Sipser Ch 1.3

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More on Pumping

Pumping Lemma (Formal)

Let L be a regular language.

Then there exists a "pumping length" p such that

For every $w \in L$ where $|w| \geq p$,
w can be split into three parts $w = xyz$ where:

1. $|y| > 0$
2. $|xy| \leq p$
3. $xy^iz \in L$ for all $i \geq 0$

General Strategy for proving L is not regular

Proof by contradiction: assume L is regular.

Then there is a pumping length p.

1. Find $w \in L$ with $|w| \geq p$

2. Show that w cannot be pumped

3. Conclude L must not have been regular

Pumping down

Claim: $L = \{0^i 1^j \mid i > j \geq 0\}$ is not regular

Proof: Assume L is regular with pumping length p

1. Find $w \in L$ with $|w| \geq p$

2. Show that w cannot be pumped
   Formally   If $w = xyz$ with $|xy| \leq p$, then...

Reusing a Proof

Pumping a language can be lots of work…

Let's try to reuse that work!

How might we show that

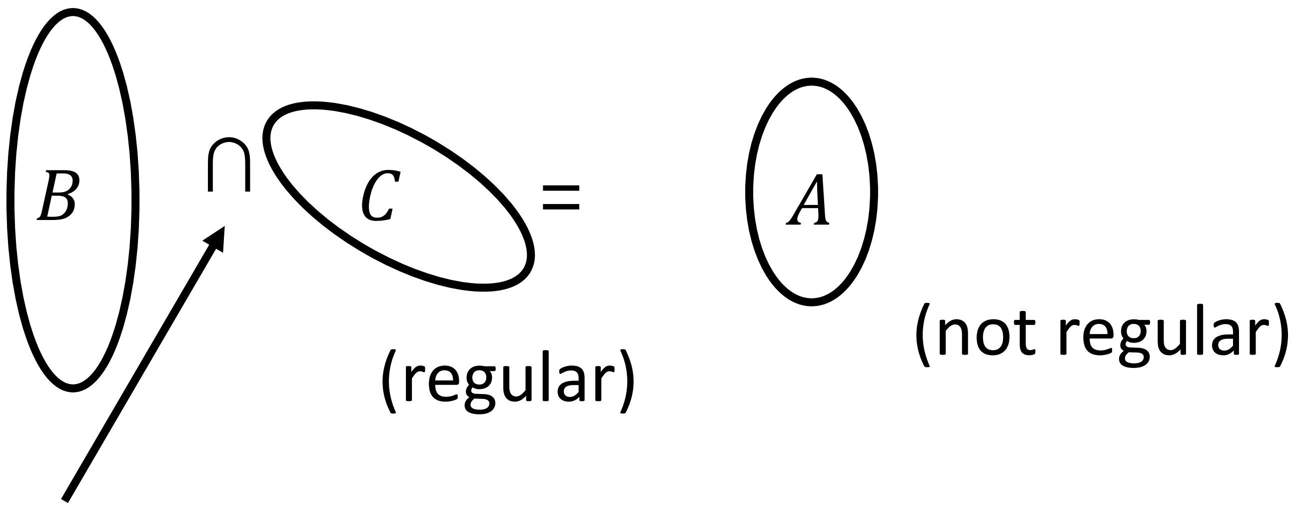
$BALANCED$ = $\{w \mid w$ has an equal # of 0s and 1s$\}$

is not regular?

$\{0^n1^n \mid n \geq 0\}$ = $BALANCED \cap \{w \mid$ all 0s in w appear before all 1s$\}$

Using Closure Properties

If A is not regular, we can show a related language B is not regular



any of $\{\circ, \cup, \cap\}$ or, for one language, $\{\neg, {}^{R}, *\}$

<u>By contradiction:</u> If B is regular, then $B \cap C$ $(= A)$ is regular.

But A is not regular so neither is B!

Example

Prove $B = \{0^i 1^j | i \neq j\}$ is not regular using nonregular language $A = \{0^n 1^n | n \geq 0\}$ and regular language

$C = \{w$ | all 0s in w appear before all 1s$\}$

Regular Expressions

Regular Expressions

- A different way of describing regular languages
- A regular expression expresses a (possibly complex) language by combining simple languages using the regular operations

"Simple" languages: $\emptyset, \{\varepsilon\}, \{a\}$ for some $a \in \Sigma$

Regular operations:

Union: $A \cup B$

Concatenation: $A \circ B = \{ab \mid a \in A, b \in B\}$

Star: $A^* = \{ a_1 a_2 \ldots a_n \mid n \geq 0$ and $a_i \in A\}$

Regular Expressions – Syntax

A regular expression R is defined recursively using the following rules:

1. ε, $\emptyset$, and a are regular expressions for every $a \in \Sigma$

2. If R_1 and R_2 are regular expressions, then so are

   $(R_1 \cup R_2)$, $(R_1 \circ R_2)$, and (R_1^*)

Examples: (over $\Sigma = \{a, b, c\}$)

$(a \circ b)$ $\quad$ $((((a \circ (b^*)) \circ c) \cup (((a^*) \circ b))^*))$ $\quad$ $(\emptyset^*)$

Regular Expressions – Semantics

$L(R)$ = the language a regular expression describes

1. $L(\emptyset) = \emptyset$
2. $L(\varepsilon) = \{\varepsilon\}$
3. $L(a) = \{a\}$ for every $a \in \Sigma$
4. $L((R_1 \cup R_2)) = L(R_1) \cup L(R_2)$
5. $L((R_1 \circ R_2)) = L(R_1) \circ L(R_2)$
6. $L\big((R_1^*)\big) = (L(R_1))^*$

Example: $L(((a^*) \circ (b^*))) =$

Simplifying Notation

- Omit $\circ$ symbol: $(ab) = (a \circ b)$

- Omit many parentheses, since union and concatenation are associative:

$$(a \cup b \cup c) = (a \cup (b \cup c)) = ((a \cup b) \cup c)$$

- Order of operations: Evaluate star, then concatenation, then union

$$ab^* \cup c = (a(b^*)) \cup c$$

Examples

Let $\Sigma = \{0, 1\}$

1. $\{w \mid w$ contains exactly one $1\}$

2. $\{w \mid w$ has length at least 3 and its third symbol is $0\}$

3. $\{w \mid$every odd position of w is $1\}$

Syntactic Sugar

- For alphabet Σ, the regex Σ represents $L(\Sigma) = \Sigma$
- For regex R, the regex $R^{+} = RR^{*}$

Not captured by regular expressions: Backreferences

Equivalence of Regular Expressions, NFAs, and DFAs

Regular Expressions Describe Regular Languages

Theorem: A language A is regular if and only if it is described by a regular expression

Theorem 1: Every regular expression has an equivalent NFA

Theorem 2: Every NFA has an equivalent regular expression

Regular expression -> NFA

Theorem 1: Every regex has an equivalent NFA

Proof: Induction on size of a regex

Base cases:

$$R = \emptyset$$

$$R = \varepsilon$$

$$R = a$$

Regular expression -> NFA

Theorem 1: Every regex has an equivalent NFA

Proof: Induction on size of a regex

Inductive step:

$$R = (R_1 \cup R_2)$$

$$R = (R_1 R_2)$$

$$R = (R_1^*)$$

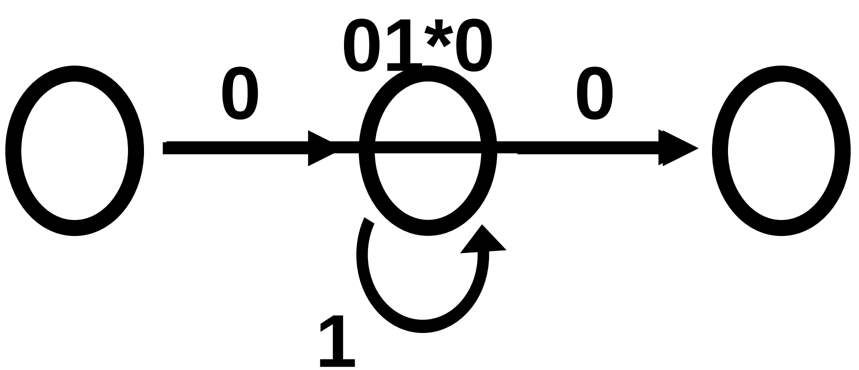
Example

Convert $(1(0 \cup 1))^*$ to an NFA

NFA -> Regular expression

Theorem 2: Every NFA has an equivalent regex

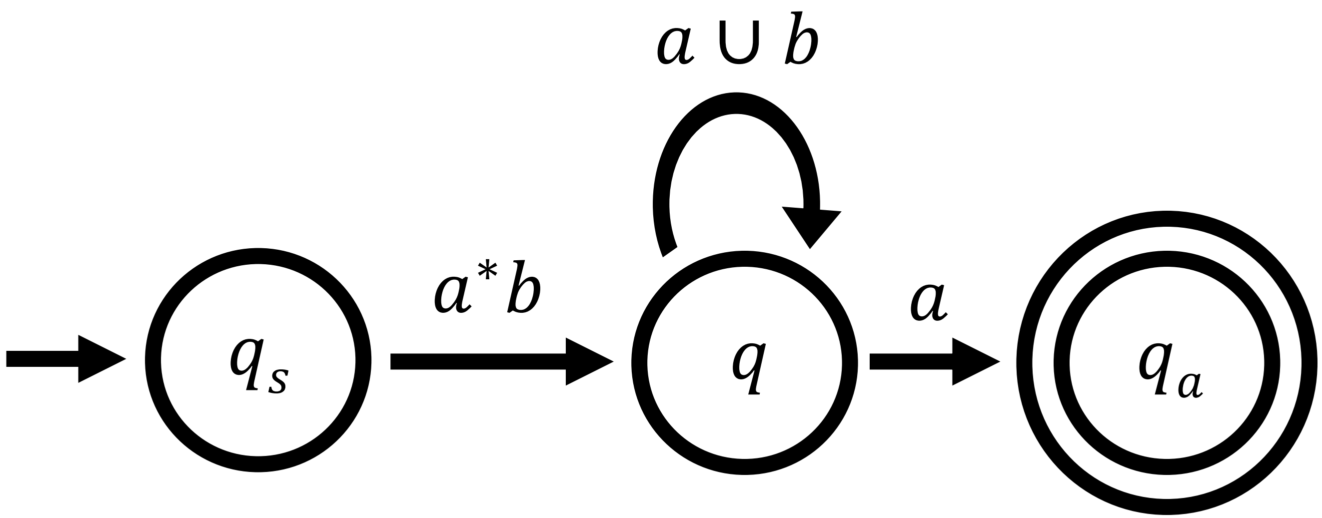
Proof idea: Simplify NFA by “ripping out” states one at a time and replacing with regexes



Generalized NFAs

- **Every transition is labeled by a regex**
- One start state with only outgoing transitions
- Only one accept state with only incoming transitions
- Start state and accept state are distinct

Generalized NFA Example

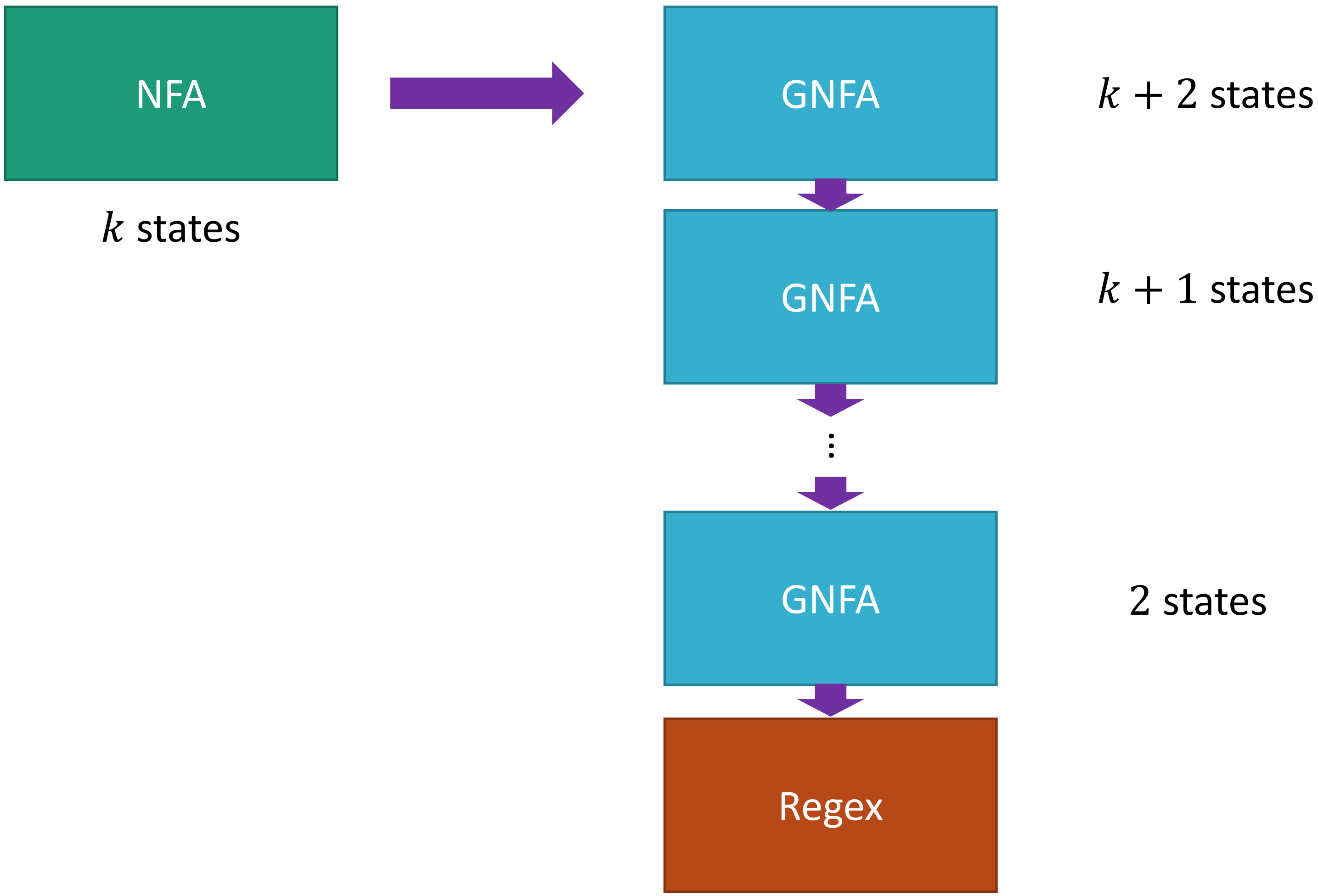


$$R(q_s, q) \ =$$

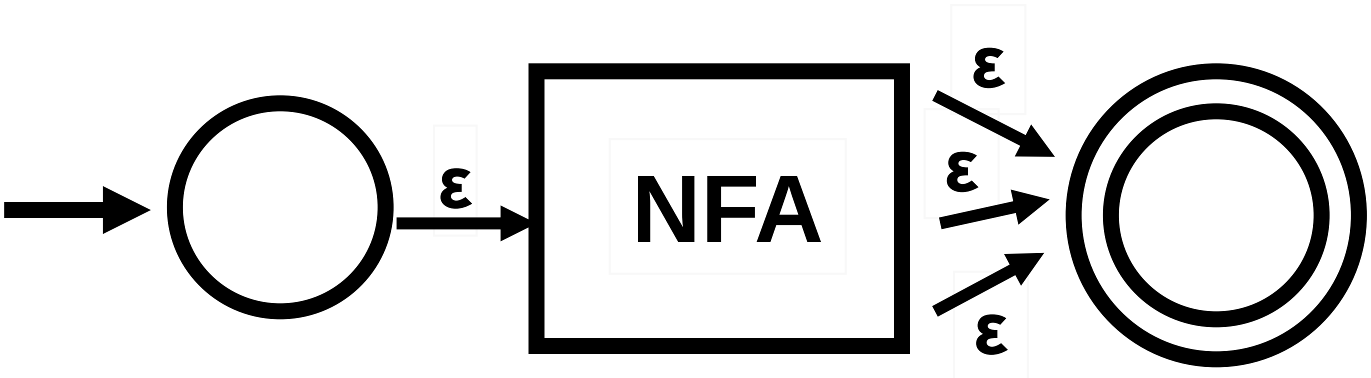
$$R(q_a, q) \ =$$

$$R(q, qs) \ =$$

NFA -> Regular expression



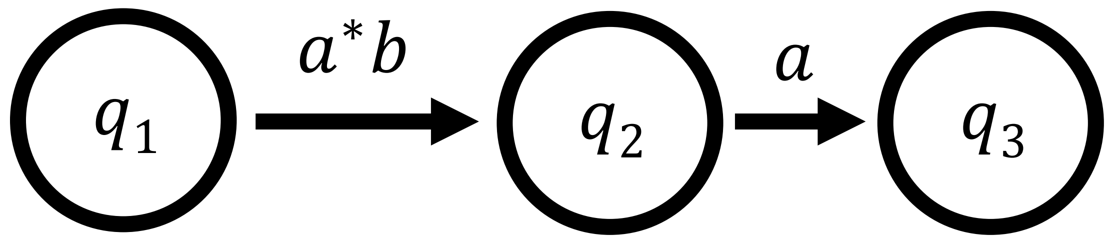
NFA -> GNFA

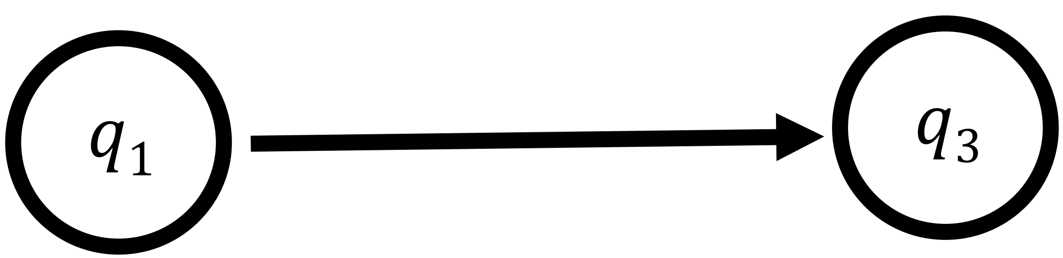


- Add a new start state with no incoming arrows.
- Make a unique accept state with no outgoing arrows.

GNFA -> Regular expression

Idea: While the machine has more than 2 states, rip one out and relabel the arrows with regexes to account for the missing state





GNFA -> Regular expression

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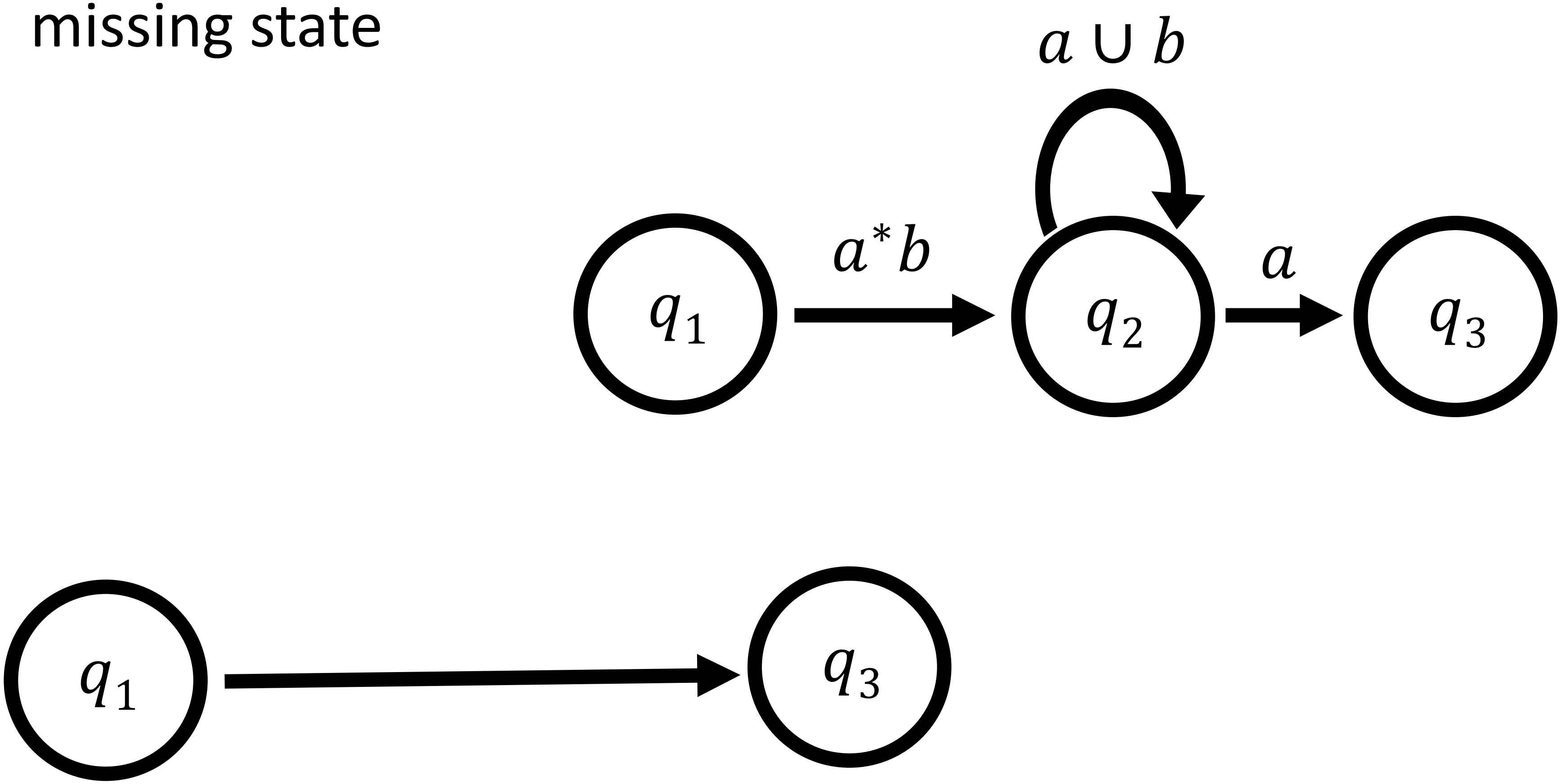

GNFA -> Regular expression

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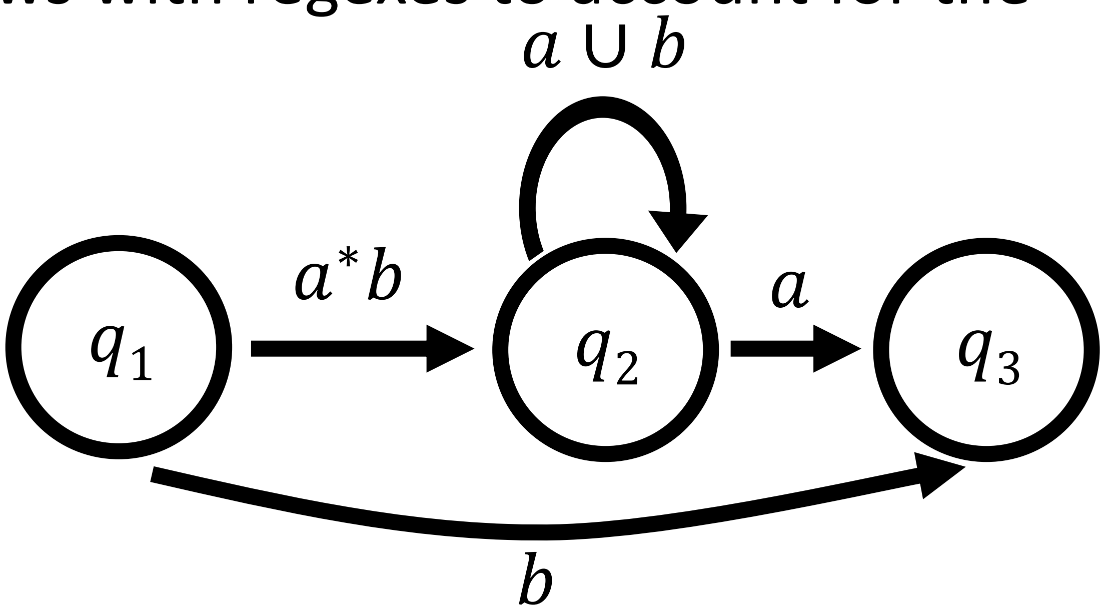

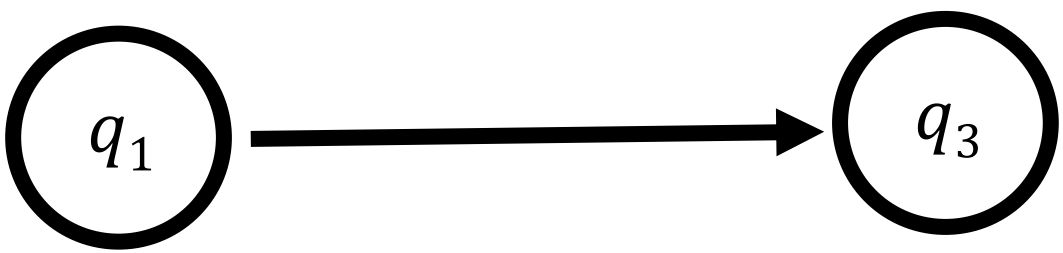

GNFA -> Regular expression

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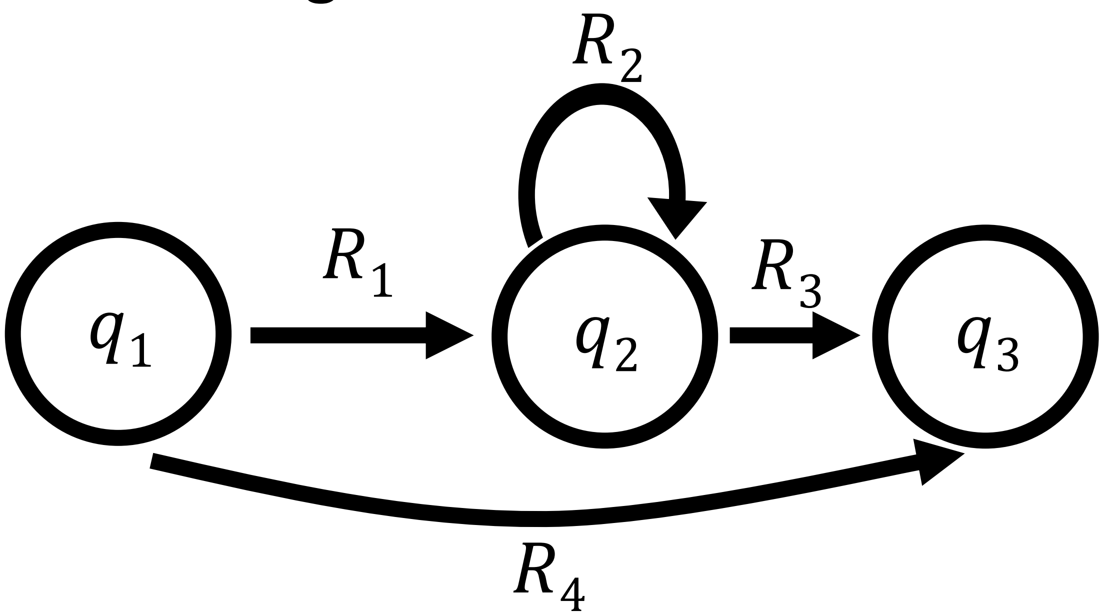

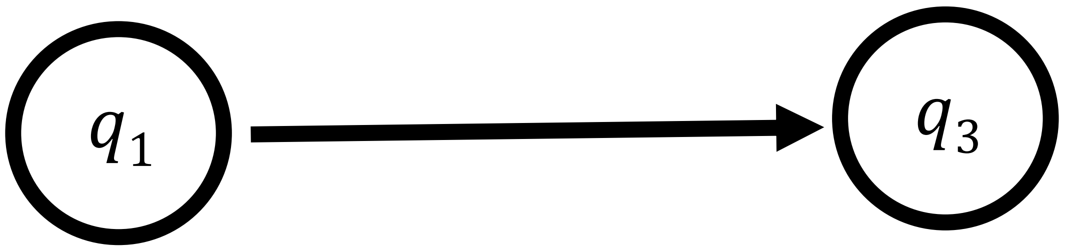

Example