BU CS 332 – Theory of Computation

Lecture 8:

- Equivalence between PDAs and CFGs
- Closure Properties

Reading:

Sipser Ch 2.2

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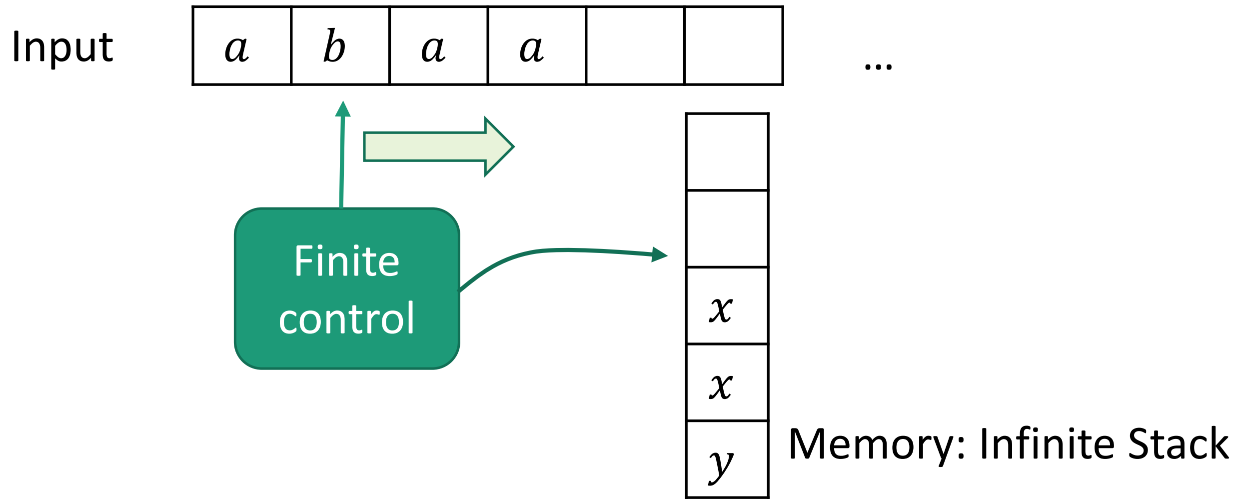
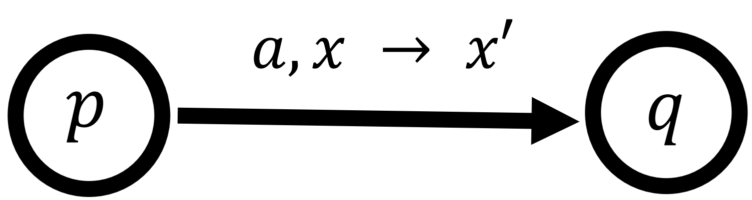
Pushdown Automaton (the idea)

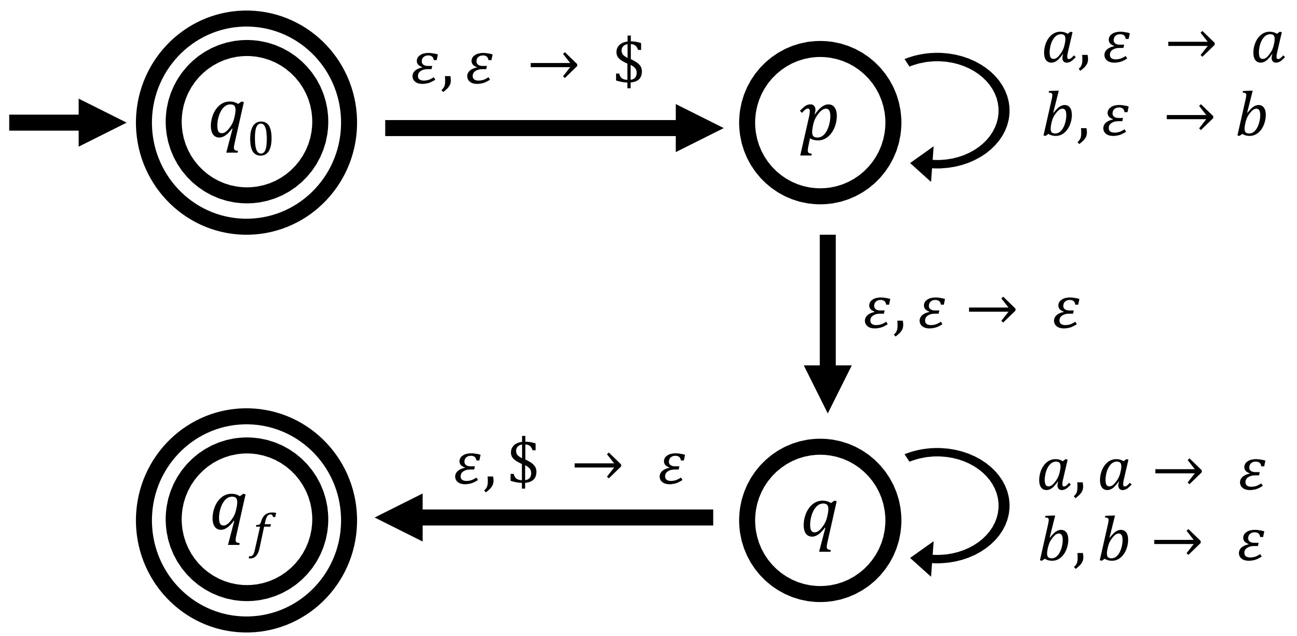
- Nondeterministic finite automaton + stack
- Stack has unlimited size, but machine can only manipulate (push, pop, read) symbol at the top



Transitions of the form:



Example: Even Palindromes



Pushdown Automaton (formal)

A PDA is a 6-tuple $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$

- Q is a finite set of states
- Σ is the input alphabet
- Γ is the stack alphabet
- $\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \rightarrow P(Q \times \Gamma_\varepsilon)$ is the transition function
- q_0 is the start state
- F is the set of final states

M ***accepts*** a string w if, starting from q_0 and an empty stack, **there exists** a path to an accept state that can be followed by reading all of w.

Example

$L = \{w|w \text{ has an equal number of } a\text{'s and } b\text{'s}\}$

Algorithmic description

Example

$L = \{w | w \text{ has an equal number of } a\text{'s and } b\text{'s}\}$

State diagram

CFGs vs. PDAs

The language $L(M)$ of a PDA M is the set of all strings it accepts.

Theorem: The class of languages recognized by PDAs is exactly the context-free languages.

Theorem 1: Every CFG has an equivalent PDA

Theorem 2: Every PDA has an equivalent CFG

CFG -> PDA

CFG -> PDA Conversion

Suppose language L is generated by CFG $G = (V, \Sigma, R, S)$

Goal: Construct a PDA $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$ recognizing L

Idea: M will guess the steps of the CFG derivation of its input w, and use its stack to check the derivation

A helpful intermediate abstraction

Generalized PDA: Can push an entire string to the stack in one move

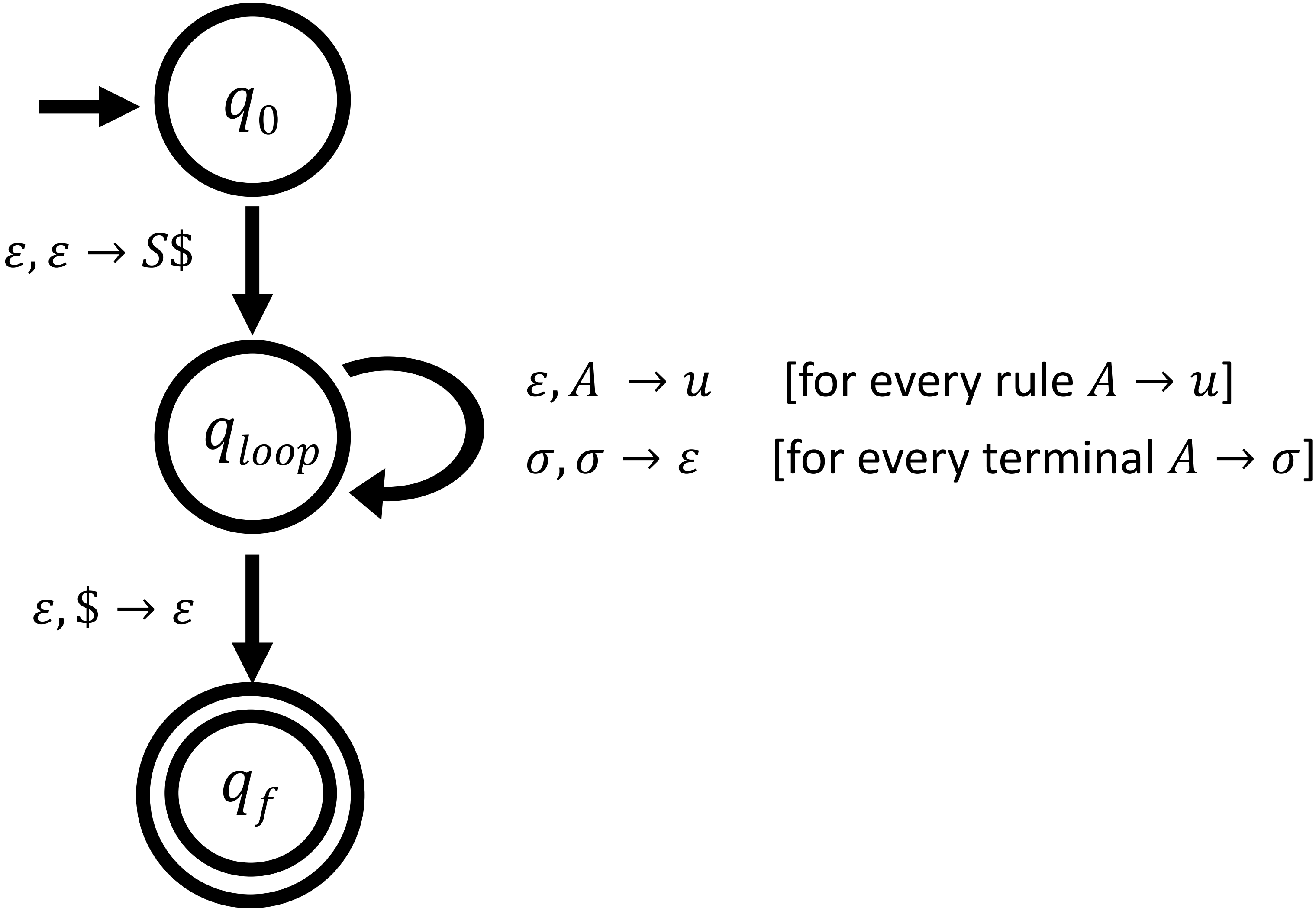
Algorithmic Description

1. Place $ and the start variable S on the stack
2. Repeat forever:
   a) If the top of the stack holds variable A:
      Nondeterministically guess a rule $(A \to u) \in R$
      Replace A with u on the stack

   b) If the top of the stack holds terminal σ:
      Pop σ and verify that it matches the next input char

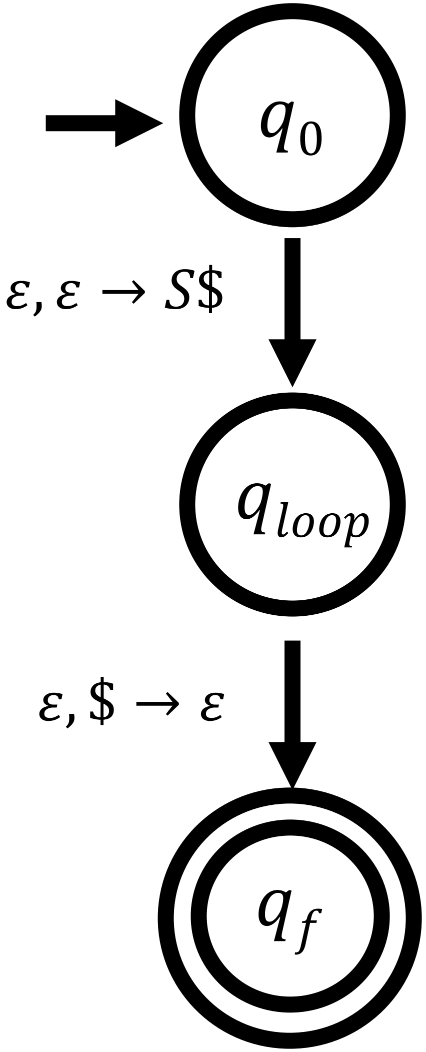
   c) If the top of the stack holds $\$$:
      Accept if the input is empty

State Diagram



Example

$$S \rightarrow aTb$$
$$T \rightarrow Ta \mid \varepsilon$$



Example

$$S \rightarrow aTb$$
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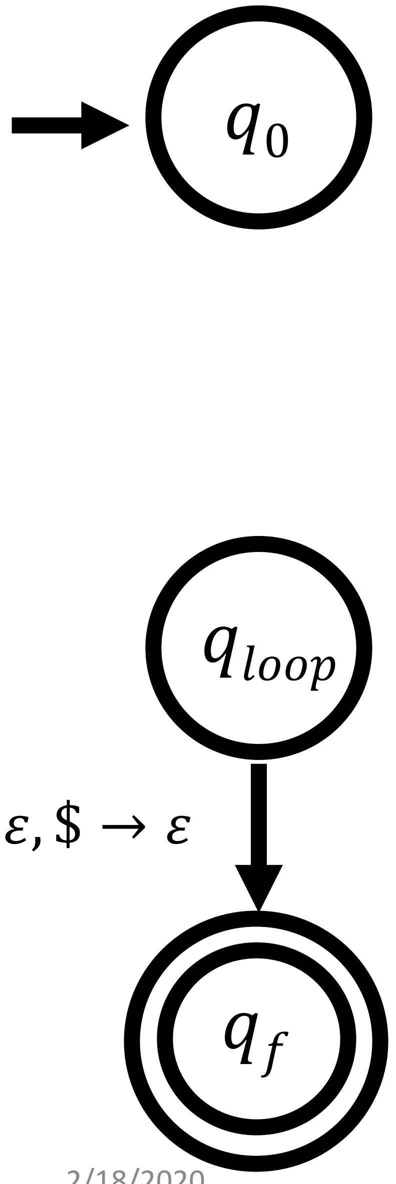


PDA -> CFG

PDA -> CFG Conversion

Theorem 2: Every PDA has an equivalent CFG

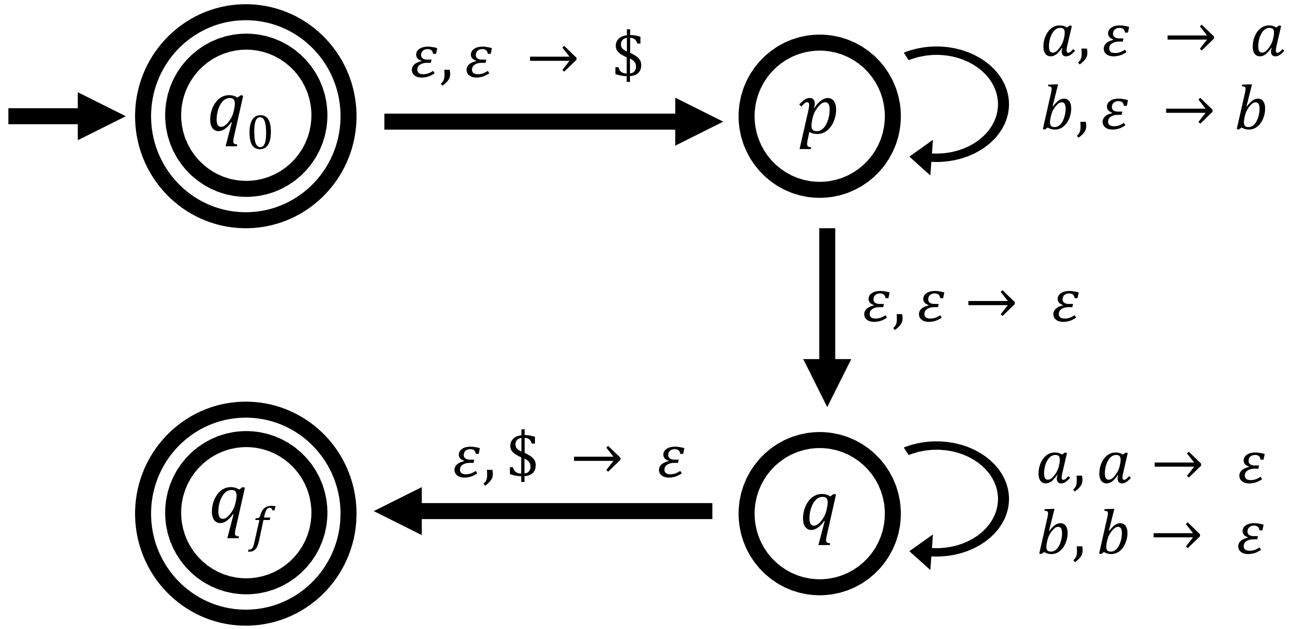
Suppose L is recognized by PDA $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$

Goal: Construct a CFG $G = (V, \Sigma, R, S)$ generating L

First simplify M so that:

1. It has a single accept state q_f
2. It empties the stack before accepting
3. Every transition either pushes a symbol or pops a symbol (but not both)

Simplification Example



Conversion Idea

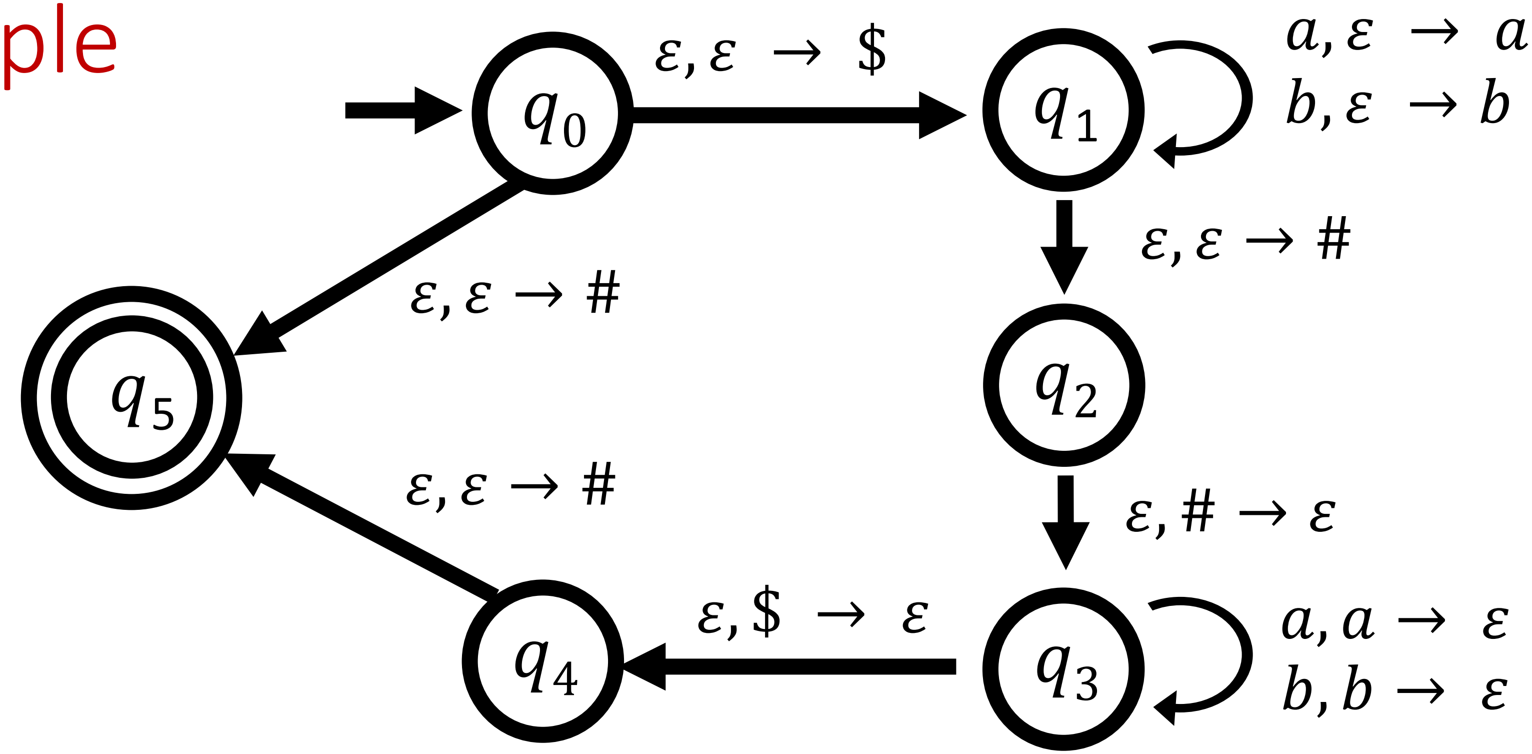
Variables: A_{pq} for every pair of states p, q in PDA M

Idea: A_{pq} generates all strings that can take M from p (with an empty stack) to q (with an empty stack)

$V =$

$S =$

Example



What strings should $A_{q_0 q_1}$ generate?

What strings should $A_{q_1 q_3}$ generate?

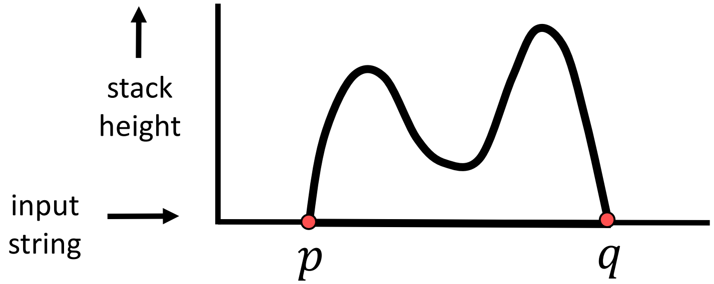
What strings should $A_{q_1 q_4}$ generate?

What rules should define A_{pq}?

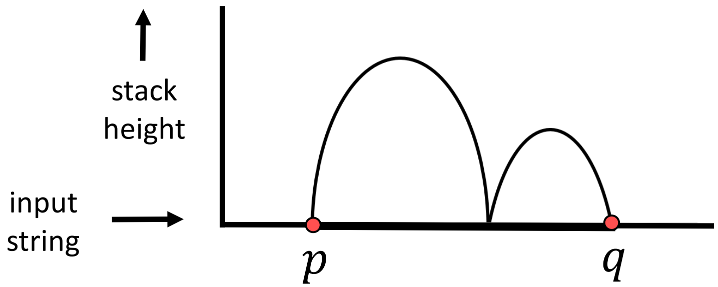
Let x be a string generated by A_{pq}

Two cases:

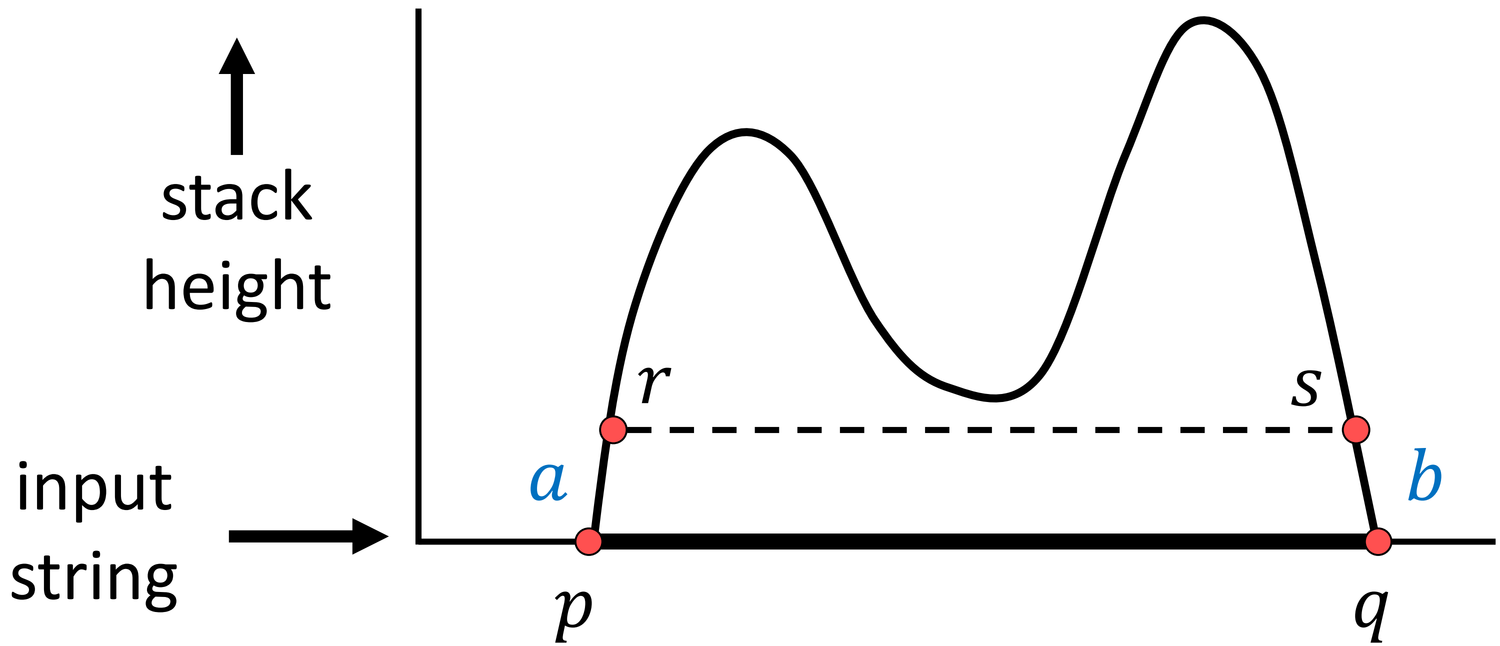
1) Stack first empties after reading all of x



2) Stack empties before reaching the end of x

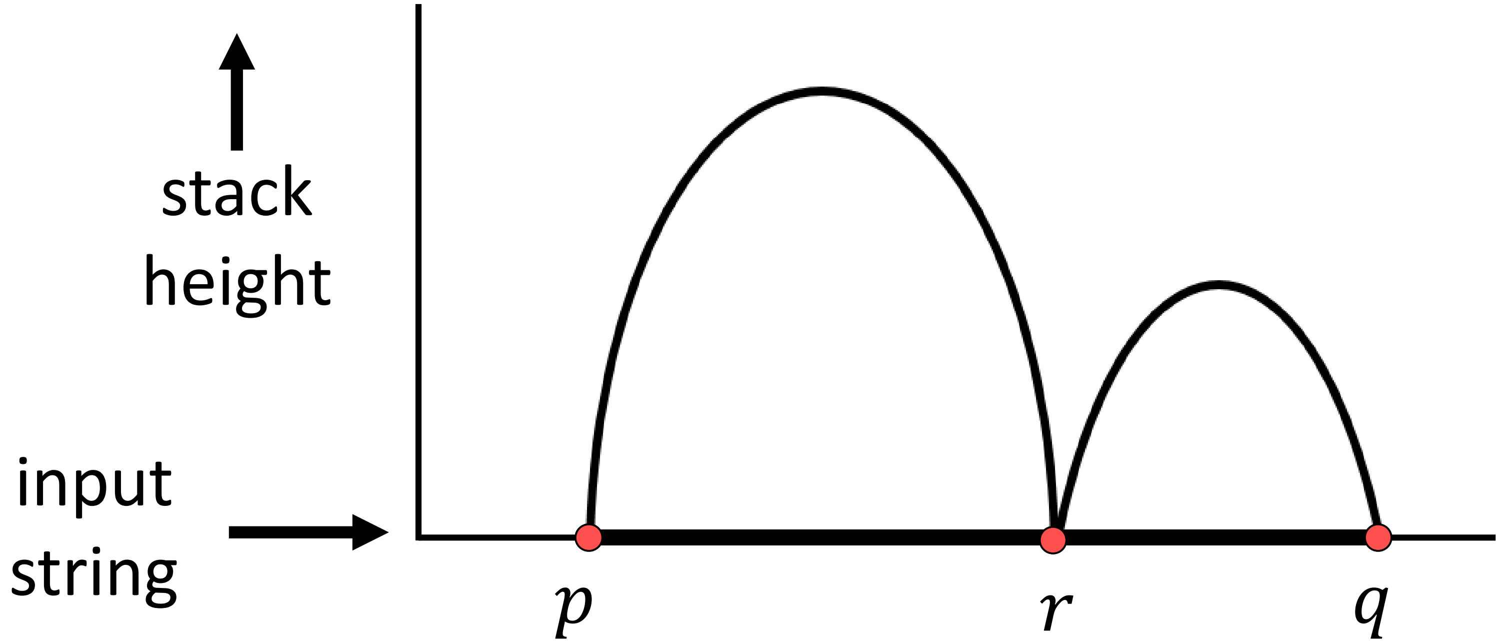


1. Stack first empties after reading all of x



Add rule $A_{pq} \to aA_{rs}b$

2. Stack empties before reaching the end of x



Add rule $A_{pq} \rightarrow A_{pr}A_{rq}$

Formal CFG Construction

$V = \{A_{pq} \mid p, q \in Q\}$

$S = A_{q_0 q_f}$

Three kinds of rules:

1. For every $p, q, r, s \in Q, \quad t \in \Gamma, \quad a, b \in \Sigma_\varepsilon$:

   If $(r, t) \in \delta(p, a, \varepsilon)$ and $(q, \varepsilon) \in \delta(s, b, t)$,

   include the rule $A_{pq} \to aA_{rs}b$

2. For every $p, q, r \in Q$, include the rule $A_{pq} \to A_{pr}A_{rq}$
3. For every $p \in Q$, include the rule $A_{pp} \to \varepsilon$

Sketch of proof that CFG generates $L(M)$

Claim: $A_{pq} \Rightarrow^* x$ if and only if x takes M from p to q, beginning and ending with empty stack

Proof idea:

$\Longrightarrow$ Induction on number of steps of derivation of x from A_{pq}

$\Longleftarrow$ Induction on number of steps of computation taking M from p to q

Closure Properties

Closure Properties

- The class of CFLs is closed under

  Union

  Concatenation

  Star

  Intersection with *regular* languages

- Beware: It is not closed under intersection or complement

  (Find counterexamples!)

Closure under union (Proof 1)

Let A be a CFL recognized by PDA M_A and let B be a CFL recognized by PDA M_B

Goal: Construct a PDA recognizing $A \cup B$

Closure under union (Proof 2)

Let A be a CFL generated by CFG G_A and let B be a CFL recognized by CFG G_B

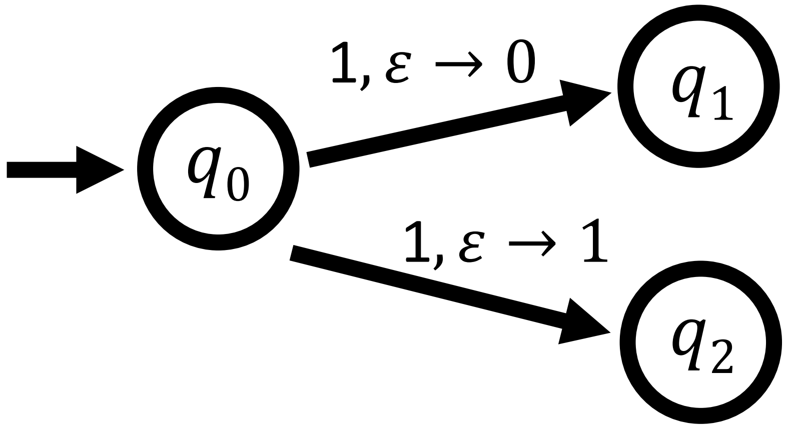
Goal: Construct a CFG G recognizing $A \cup B$

$G_A = (V_A, \Sigma_A, R_A, S_A)$

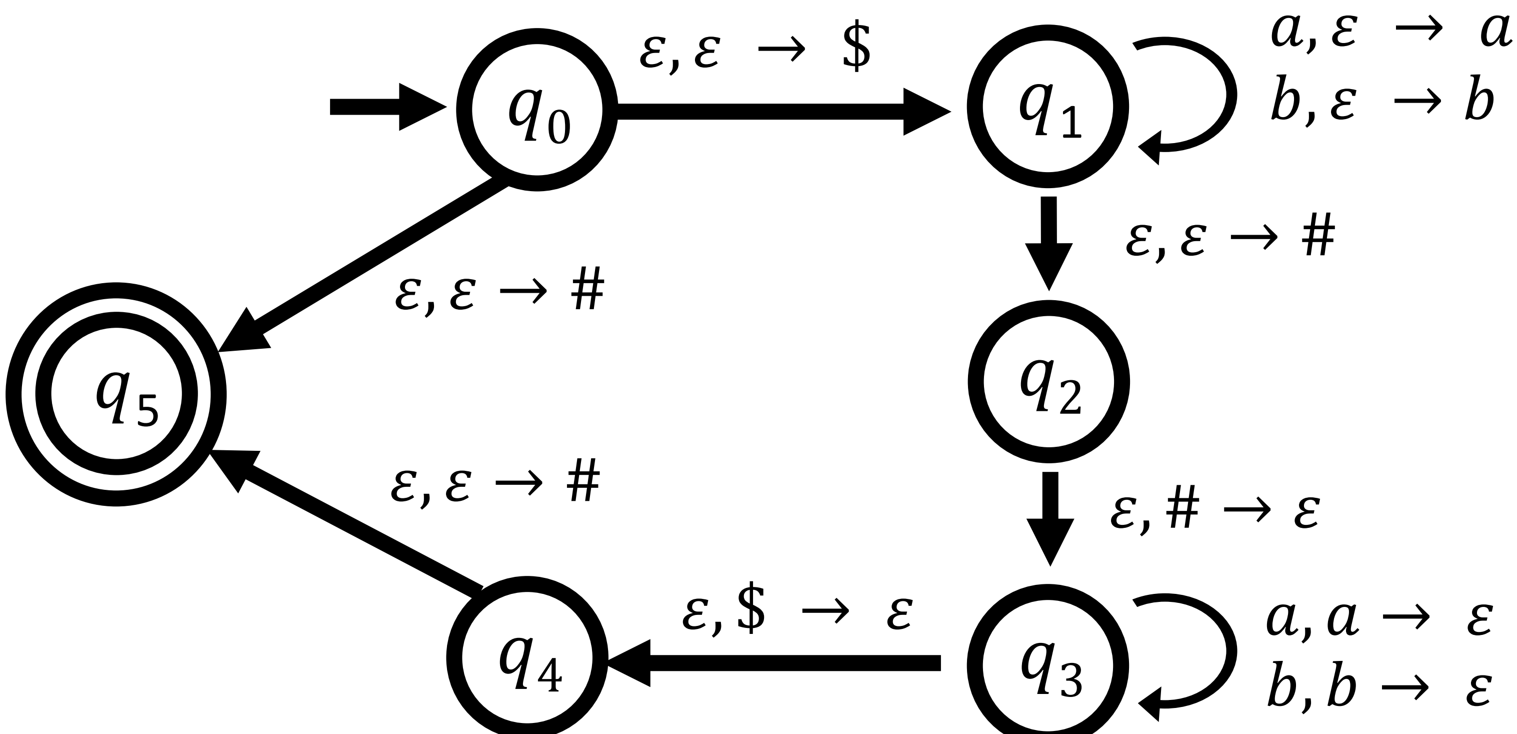
$G_B = (V_B, \Sigma_B, R_B, S_B)$

Relabel variables so V_A and V_B are disjoint

Let $G = (V, \Sigma, R, S)$



$1, \varepsilon \to 0$
q_1
q_0
$1, \varepsilon \to 1$
q_2



$\varepsilon, \varepsilon \to \$$
q_0
q_1
$a, \varepsilon \to a$
$b, \varepsilon \to b$
$\varepsilon, \varepsilon \to \#$
$\varepsilon, \varepsilon \to \#$
q_5
q_2
$\varepsilon, \varepsilon \to \#$
$\varepsilon, \# \to \varepsilon$
$\varepsilon, \$ \to \varepsilon$
q_4
q_3
$a, a \to \varepsilon$
$b, b \to \varepsilon$