BU CS 332 – Theory of Computation

Lecture 11:

- TM Variants
- Closure Properties

Reading:

Sipser Ch 3.2

Mark Bun

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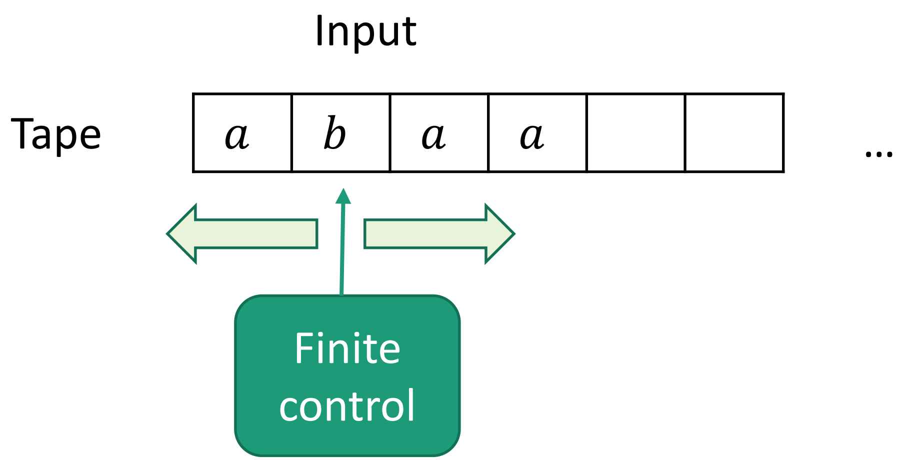
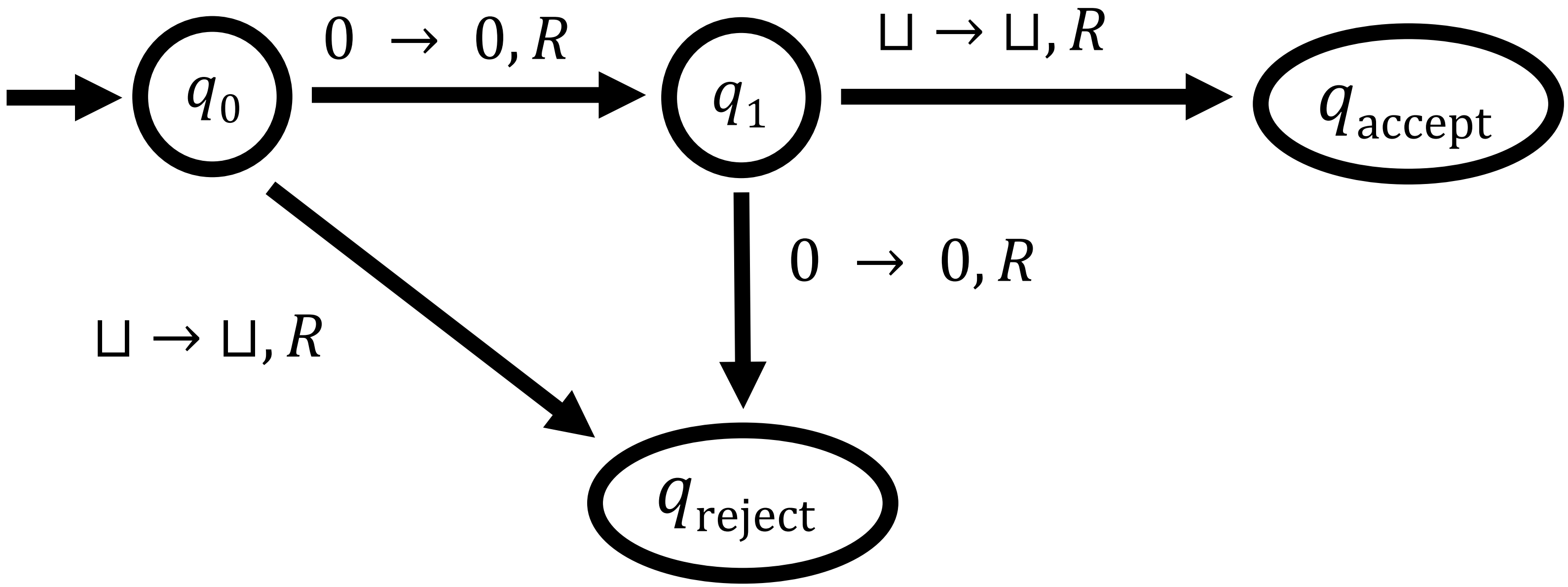
The Basic Turing Machine (TM)



- Input is written on an infinitely long tape
- Head can both read and write, and move in both directions
- Computation halts when control reaches "accept" or "reject" state

Example



q_0
$0 \rightarrow 0, R$
q_1
$\sqcup \rightarrow \sqcup, R$
q_{accept}
$0 \rightarrow 0, R$
$\sqcup \rightarrow \sqcup, R$
q_{reject}

Formal Definition of a TM

A TM is a 7-tuple $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$

- Q is a finite set of states
- Σ is the input alphabet (does **not** include $\sqcup$)
- Γ is the tape alphabet (contains $\sqcup$ and Σ)
- δ is the transition function

…more on this later

- $q_0 \in Q$ is the start state
- $q_{\text{accept}} \in Q$ is the accept state
- $q_{\text{reject}} \in Q$ is the reject state ($q_{\text{reject}} \neq q_{\text{accept}}$)

TM Transition Function

$$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$$

L means "move left" and R means "move right"

$\delta(p, a) = (q, b, R)$ means:

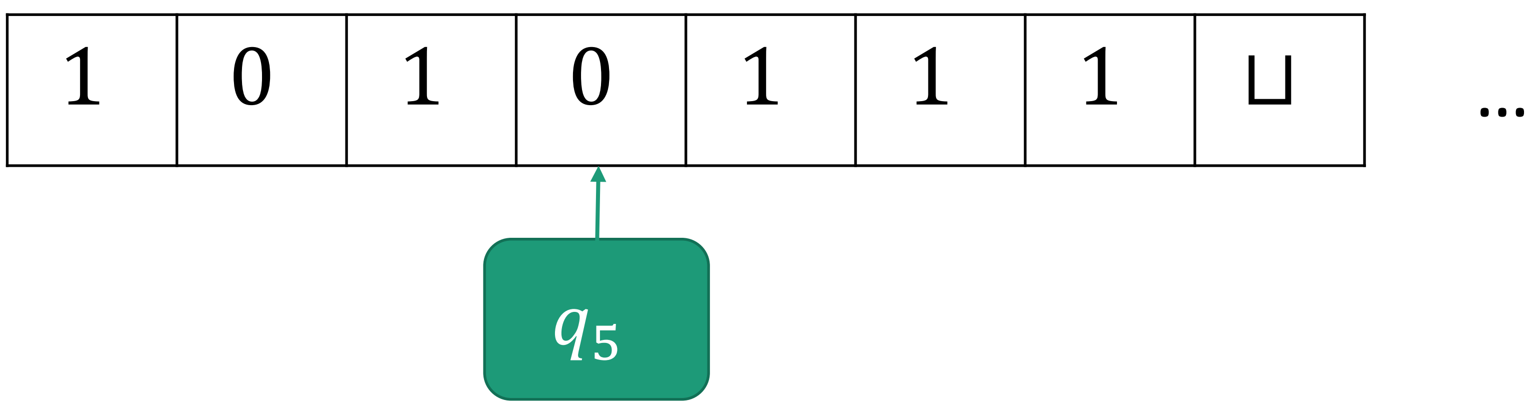
- Replace a with b in current cell
- Transition from state p to state q
- Move tape head right

$\delta(p, a) = (q, b, L)$ means:

- Replace a with b in current cell
- Transition from state p to state q
- Move tape head left UNLESS we are at left end of tape, in which case don't move

Configuration of a TM

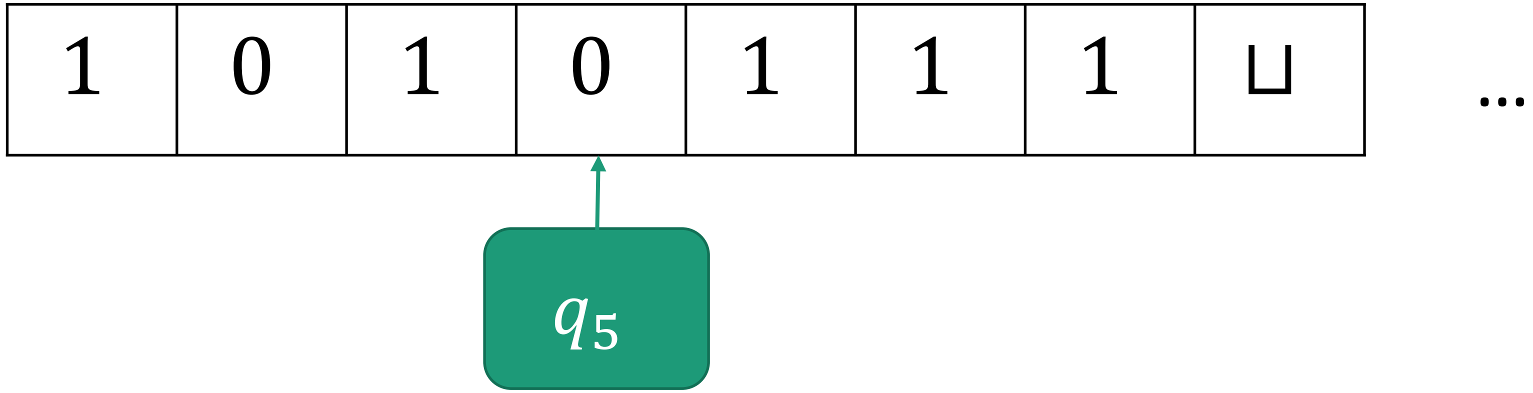
A string with captures the state of a TM together with the contents of the tape



Configuration of a TM: Formally

A configuration is a string uqv where $q \in Q$ and $u, v \in \Gamma^*$

- Tape contents = uv (followed by blanks ⊔)
- Current state = q
- Tape head on first symbol of v



How a TM Computes

Start configuration: $q_0 w$

One step of computation:

- $ua\ q\ bv$ yields $uac\ q'\ v$ if $\delta(q, b) = (q', c, R)$
- $ua\ q\ bv$ yields $u\ q'\ acv$ if $\delta(q, b) = (q', c, L)$
- $q\ bv$ yields $q'\ cv$ if $\delta(q, b) = (q', c, L)$

Accepting configuration: q = q_{accept}

Rejecting configuration: q = q_{reject}

How a TM Computes

M accepts input w if there is a sequence of configurations $C_1, \ldots, C_k$ such that:

- $C_1 = q_0 w$
- C_i yields C_{i+1} for every i
- C_k is an accepting configuration

$L(M) =$ the set of all strings w which M accepts

A is Turing-recognizable if $A = L(M)$ for some TM M:

- $w \in A \implies M$ halts on w in state q_{accept}
- $w \notin A \implies M$ halts on w in state q_{reject} OR
  M runs forever

Recognizers vs. Deciders

$L(M)$ $=$ the set of all strings w which M accepts

A is Turing-recognizable if $A = L(M)$ for some TM M:

- $w \in A \implies M$ halts on w in state q_{accept}
- $w \notin A \implies M$ halts on w in state q_{reject} OR
  M runs forever

A is (Turing-)decidable if $A = L(M)$ for some TM M
which halts on every input

- $w \in A \implies M$ halts on w in state q_{accept}
- $w \notin A \implies M$ halts on w in state q_{reject}

Back to Hilbert's Tenth Problem

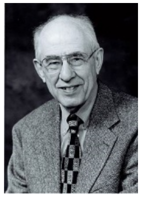
Computational Problem: Given a Diophantine equation, does it have a solution over the integers?

$L =$

- L is Turing-recognizable

- L is not decidable (1949-70)






TM Variants

How Robust is the TM Model?

Does changing the model result in different languages being recognizable / decidable?

So far we've seen…

- We can require that FAs/PDAs have a single accept state
- (CFGs can always be put in Chomsky Normal Form)
- Adding nondeterminism does not change the languages recognized by finite automata

Turing machines have an astonishing level of robustness

Extensions that do not increase the power of the TM model

- TMs that are allowed to "stay put" instead of moving left or right

$$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R, S\}$$

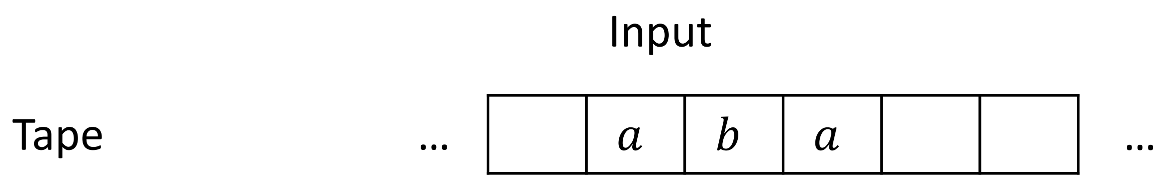
Proof that TMs with "stay put" are no more powerful:

Simulation: Convert any TM M with "stay put" into an equivalent TM M' without

Replace every "stay put" instruction in M with a move right instruction, followed by a move left instruction in M'

Extensions that do not increase the power of the TM model

- TMs with a 2-way infinite tape, unbounded left to right



Proof that TMs with 2-way infinite tapes are no more powerful:

Simulation: Convert any TM M with 2-way infinite tape into a 1-way infinite TM M' with a "two-track tape"

Formalizing the Simulation

$M' = (Q', \Sigma, \Gamma', \delta', q_0', q'_{\mathrm{accept}}, q'_{\mathrm{reject}})$

New tape alphabet: $\Gamma' = (\Gamma \times \Gamma) \cup \{\$\}$

New state set: $Q' = Q \times \{+, -\}$

$(q, -)$ means "q, working on upper track"

$(q, +)$ means "q, working on lower track"

New transitions:

If $\delta(p, a_-) = (q, b, L)$, let $\delta'\big((p, -), (a_-, a_+)\big) = ((q, -), (b, a_+), R)$

Also need new transitions for moving right, lower track, hitting \$, initializing input into 2-track format

TMs are equivalent to…

- TMs with "stay put"
- TMs with 2-way infinite tapes
- Multi-tape TMs
- Nondeterministic TMs
- Random access TMs
- Enumerators
- Finite automata with access to an unbounded queue = 2-stack PDAs
- Primitive recursive functions
- Cellular automata

…

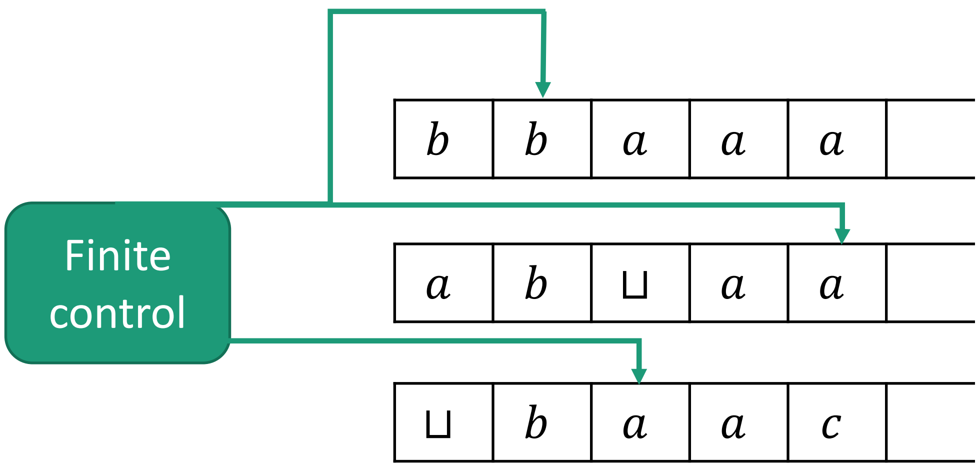
Church-Turing Thesis

The equivalence of these models is a mathematical theorem

Church-Turing *Thesis*: Each of these models captures our intuitive notion of algorithms

The Church-Turing Thesis is not a mathematical statement!

Multi-Tape TMs

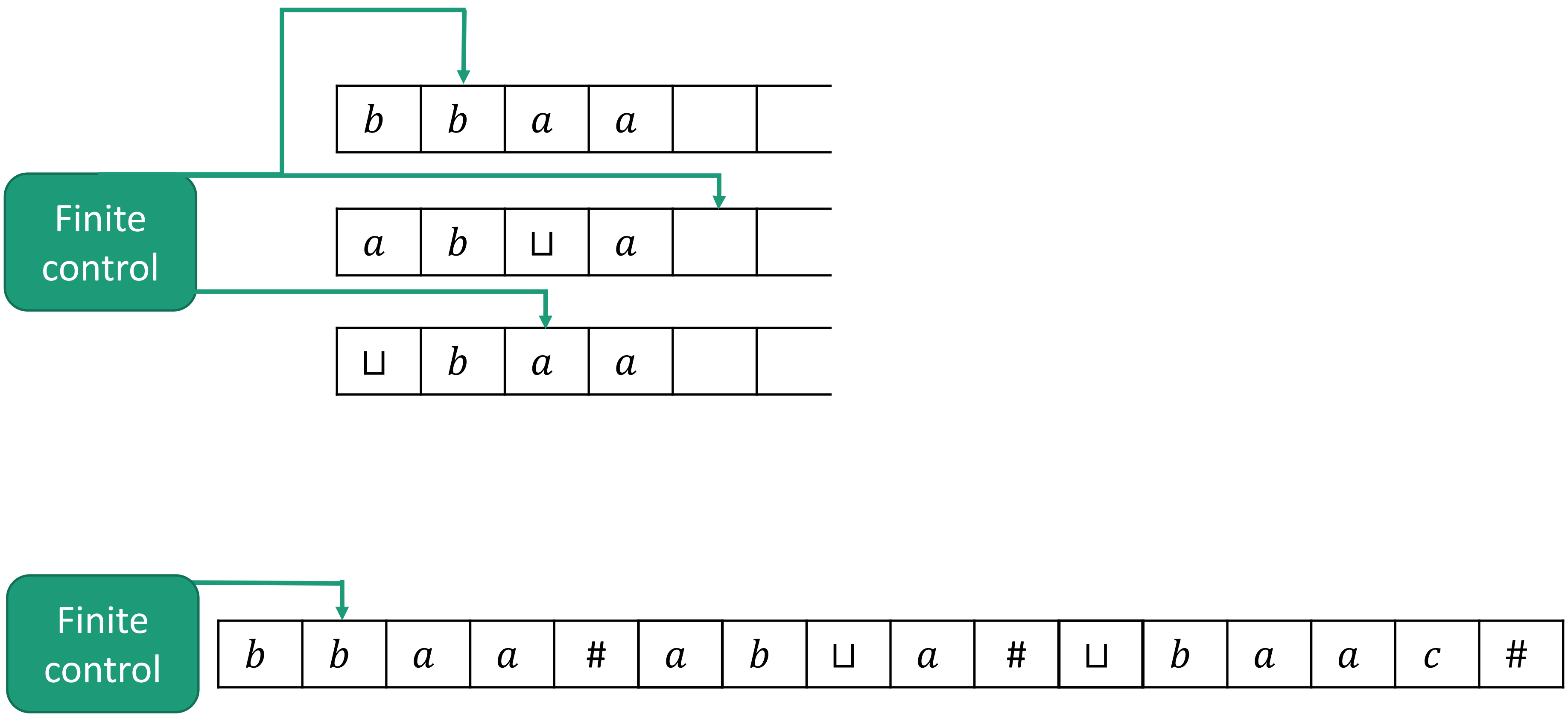


Fixed number of tapes k (can’t change during computation)

Transition function $\delta : Q \times \Gamma^k \to Q \times \Gamma^k \times \{L, R, S\}^k$

Multi-Tape TMs are Equivalent to Single-Tape TMs

Theorem: Every k-tape TM M with can be simulated by an equivalent single-tape TM M'



Simulating Multiple Tapes

Implementation-Level Description

On input $w = w_1 w_2 \ldots w_n$

1. Format tape into $\# \dot{w}_1 w_2 \ldots w_n \# \dot{\sqcup} \# \dot{\sqcup} \# \ldots \#$
2. For each move of M:

    Scan left-to-right, finding current symbols

    Scan left-to-right, writing new symbols,

    Scan left-to-right, moving each tape head

    If a tape head goes off the right end, insert blank

    If a tape head goes off left end, move back right

Why are Multi-Tape TMs Helpful?

To show a language is Turing-recognizable or decidable, suffices to construct a multi-tape TM

Very helpful for proving closure properties

Ex. Closure of recognizable languages under union. Suppose M_1 is a single-tape TM recognizing L_1, M_2 is a single-tape TM recognizing L_2

Non-deterministic TMs

At any point in computation, may non-deterministically branch. Accepts iff there exists an accepting branch.

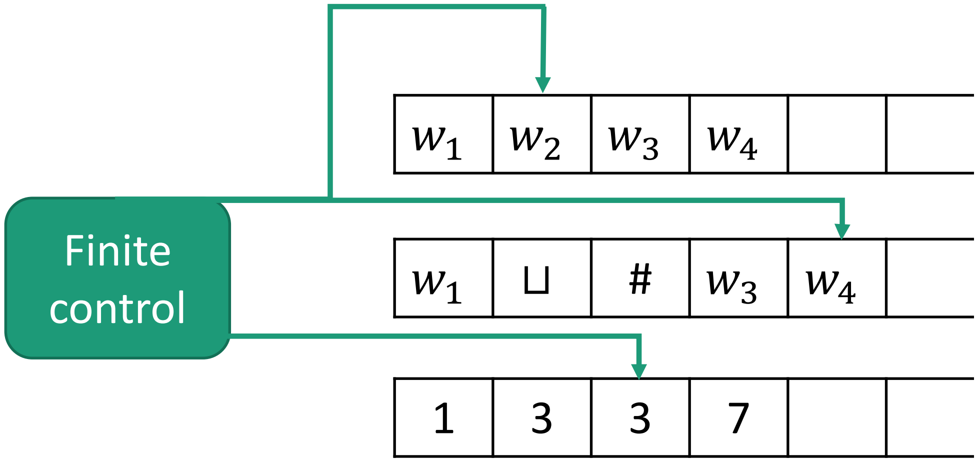
Transition function $\delta : Q \times \Gamma \rightarrow P(Q \times \Gamma \times \{L, R, S\})$

Ex. NTM for $\{w \mid w$ is a binary number representing the product of two positive integers $a, b\}$

Non-deterministic TMs

Theorem: Every nondeterministic TM has an equivalent deterministic TM

Proof idea: Simulate an NTM N using a 3-tape TM



Input w to N (read-only)

Simulation tape (run N on w using nondeterministic choices from tape 3)

Address in computation tree