BU CS 332 – Theory of Computation

Lecture 15:

- Undecidable and Unrecognizable Languages
- Reductions

Reading:

Sipser Ch 4.2, 5.1

- Ask questions about HW after class today
- HW 6 released, due Mon. March 30
- Midterm 2: 24-hr take-home exam, released 2:30 4/1, due (on Gradescope) @ 2:30 4/2

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How can we compare sizes of infinite sets?

Definition: Two sets have the same size if there is a correspondence (bijection) between them

A set is countable if

- it is a finite set, or
- it has the same size as $\mathbb{N}$, the set of natural numbers

A general theorem about set sizes

Theorem: Let X be a set. Then the power set $\underbrace{P(X)}_{\text{set of all subsets of } X}$ does **not** have the same size as X.

Proof: Assume for the sake of contradiction that there is a correspondence $f: X \rightarrow P(X)$

<u>Goal:</u> Use diagonalization to construct a set $S \in P(X)$ that cannot be the output $f(x)$ for any $x \in X$

Undecidable Languages

Problems in language theory

Acceptance problem	A_{DFA} decidable	A_{CFG} decidable	A_{TM} ?
Emptiness testing	E_{DFA} decidable	E_{CFG} decidable	E_{TM} ?
Equality	EQ_{DFA} decidable	EQ_{CFG} ?	EQ_{TM} ?

Undecidability

These natural computational questions about computational models are **undecidable**

I.e., computers cannot solve these problems no matter how much time they are given

An existential proof

Theorem: There exists an undecidable language over $\{0,1\}$

Proof: "Counting argument"

A simplifying assumption: Every string in $\{0,1\}^*$ is the encoding $\langle M \rangle$ of some Turing machine M

If s is not the valid encoding of some M, declare to be the encoding of the TM that always accepts

Set of all Turing machines: $X = \{0,1\}^*$

Set of all languages over $\{0,1\}$: all subsets of $\{0,1\}^*$ $= P(X)$

$P(\{0,1\}^*)$

There are more languages than there are TM deciders!

An existential proof

Theorem: There exists an unrecognizable language over $\{0,1\}$

Proof:

A simplifying assumption: Every string in $\{0,1\}^*$ is the encoding $\langle M \rangle$ of some Turing machine M

Set of all Turing machines: $X = \{0,1\}^*$

Set of all languages over $\{0,1\}$: all subsets of $\{0,1\}^*$ $= P(X)$

There are more languages than there are TM recognizers!

An explicit undecidable language

TM M						
M_1						
M_2						
M_3						
M_4						
$\vdots$						

An explicit undecidable language

TM M	$M(\langle M_1\rangle)$? (M_1)	$M(\langle M_2\rangle)$? (M_2)	$M(\langle M_3\rangle)$? (M_3)	$M(\langle M_4\rangle)$? (M_4)		$D(\langle D\rangle)$?
M_1	~~Y~~ N	N	Y	Y	…	
M_2	N	~~N~~ Y	Y	Y		
M_3	Y	Y	~~Y~~ N	N		
M_4	N	N	Y	~~N~~ Y		
⋮					⋱	
D						~~Y~~ N ~~N~~ Y

Cell in row i, column j = $\begin{cases} Y & \text{if } M_i \text{ accepts on input } \langle M_j\rangle \\ N & \text{if } M_i \text{ rejects or loops forever on input } \langle M_j\rangle \end{cases}$

$L = \{\langle M\rangle \mid M$ is a TM that does not accept on input $\langle M\rangle\}$ $= \{\langle M\rangle \mid$ flipped diagonal answer is "Y"$\}$

Suppose D decides L

An explicit undecidable language

Theorem: $L = \{\langle M \rangle \mid M$ is a TM that does not accept on input $\langle M \rangle\}$ is undecidable ① $\langle M \rangle \in L \iff M(\langle M \rangle)$ does not accept

Proof: Suppose for contradiction, that D decides L

D decides L: $\langle M \rangle \in L \iff D(\langle M \rangle)$ accepts ②

Claim: $D(\langle D \rangle)$ is not well-defined

$D(\langle D \rangle)$ accepts $\Rightarrow \langle D \rangle \in L$ (2) $\Rightarrow D(\langle D \rangle)$ does not accept (1)

$D(\langle D \rangle)$ does not accept $\Rightarrow \langle D \rangle \notin L$ (2) $\Rightarrow D(\langle D \rangle)$ accepts (1)

$\times$

"self-acceptance problem"

Corollary: $SA_{\mathrm{TM}} = \bar{L} = \{\langle M \rangle \mid M$ is a TM that accepts on input $\langle M \rangle\}$ is undecidable (because decidable lang. closed under complement)

A more useful undecidable language

$A_{\mathrm{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM that accepts input } w\}$

Theorem: A_{TM} is undecidable

But first: A_{TM} *is* Turing-recognizable

The following "universal TM" U recognizes A_{TM}

On input $\langle M, w\rangle$:

1. Simulate running M on input w
2. If M accepts, accept. If M rejects, reject.

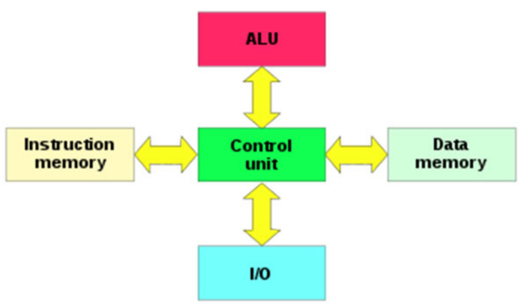
Not a decider because M may loop forever on w

More on the Universal TM

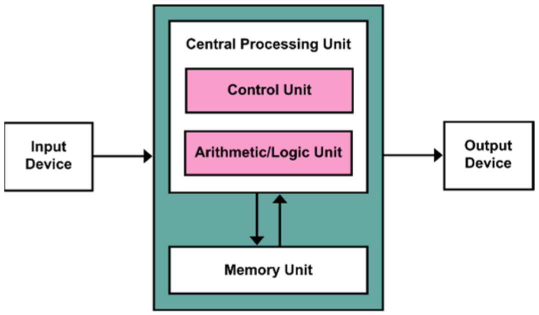
"It is possible to invent a single machine which can be used to compute any computable sequence. If this machine **U** is supplied with a tape on the beginning of which is written the S.D ["standard description"] of some computing machine **M**, then **U** will compute the same sequence as **M**."

- Turing, "On Computable Numbers…" 1936

- Foreshadowed general-purpose programmable computers
- No need for specialized hardware: Virtual machines as software



Harvard architecture:
Separate instruction and data pathways



von Neumann architecture:
Programs can be treated as data

A more useful undecidable language

$A_{\mathrm{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM that accepts input } w\}$

Theorem: A_{TM} is undecidable

Proof: Assume for the sake of contradiction that TM H decides A_{TM}:

$$H(\langle M, w\rangle) = \begin{cases} \text{accept} & \text{if } M \text{ accepts } w \\ \text{reject} & \text{if } M \text{ does not accept } w \end{cases}$$

M rejects w or loops forever

Idea: Show that H can be used to decide the (undecidable) language SA_{TM} -- a contradiction.

A more useful undecidable language

$A_{\mathrm{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM that accepts input } w\}$

Suppose H decides A_{TM}

$\langle M\rangle \in SA_{TM} \Rightarrow M(\langle M\rangle)$ accept
$\Rightarrow H(\langle M, \langle M\rangle\rangle)$ accepts
$\Rightarrow D(\langle M\rangle)$ accepts.

$\langle M\rangle \notin SA_{TM} \Rightarrow M(\langle M\rangle)$ does not accept
$\Rightarrow H(\langle M, \langle M\rangle\rangle)$ rejects
$\Rightarrow D(\langle M\rangle)$ rejects

Consider the following TM D.

On input $\langle M\rangle$ where M is a TM:

1. Run H on input $\langle M, \langle M\rangle\rangle$
2. If H accepts, accept. If H rejects, reject.

Claim: D decides

$$SA_{\mathrm{TM}} = \{\langle M\rangle \mid M \text{ is a TM that accepts on input } \langle M\rangle\}$$

...but this language is undecidable $\times$

Unrecognizable Languages

Theorem: A language L is decidable if and only if L and $\bar{L}$ are both Turing-recognizable. (L undecidable + L recognizable $\Rightarrow$ $\bar{L}$ not recognizable)

Proof:

$\Rightarrow$ L decidable $\Rightarrow$ L recognizable
$\Rightarrow$ $\bar{L}$ decidable (closure) $\Rightarrow$ $\bar{L}$ recognizable

$\Leftarrow$ $L, \bar{L}$ recognizable, recognized by TMs M_1, M_2 respectively

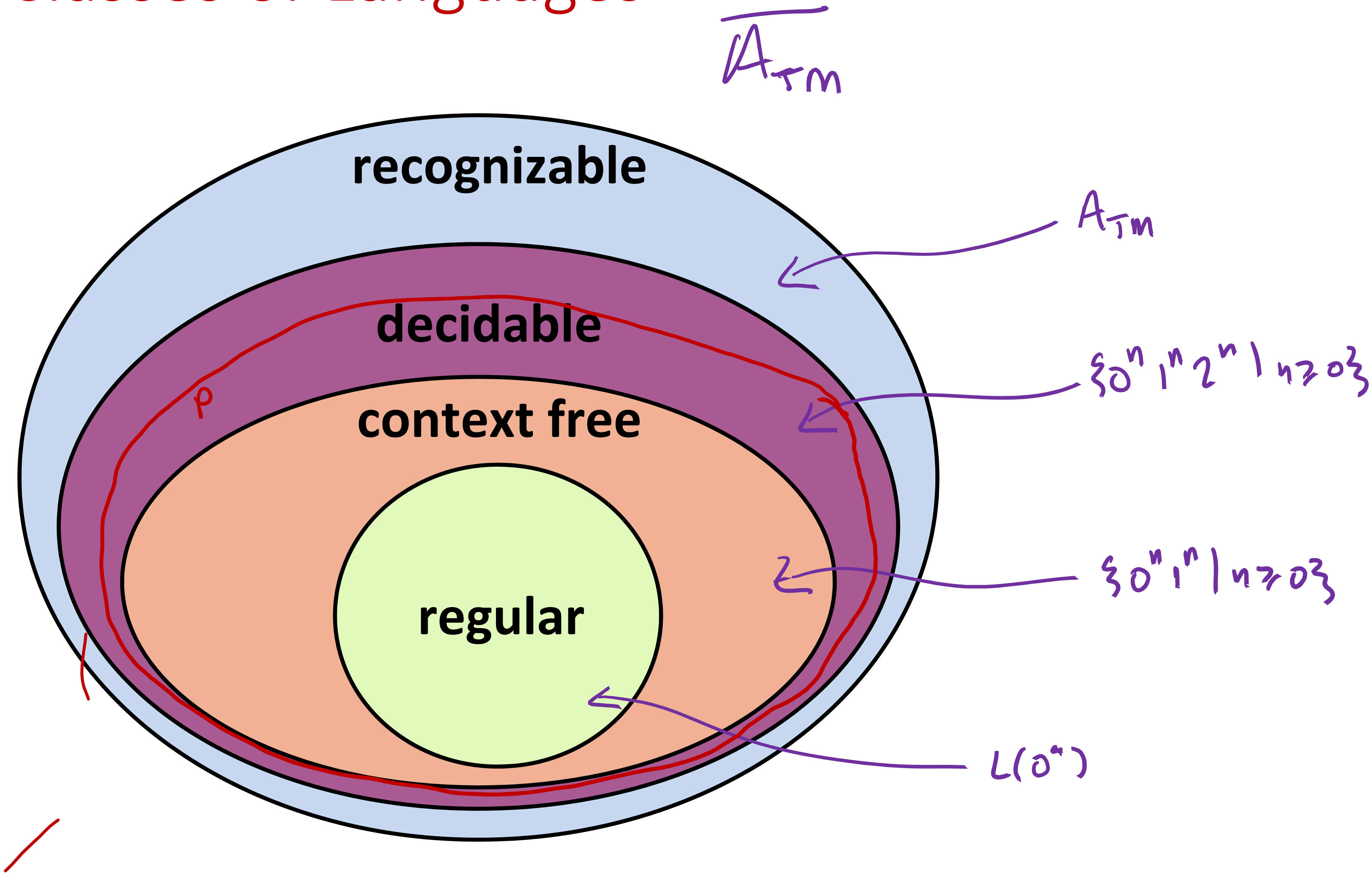
Construct decider for L:

D = " On input w:
1) Run M_1 and M_2 in parallel on w
2) If M_1 accepts, accept
If M_2 accepts, reject "

$w \in L \Rightarrow M_1$ accepts w
$\Rightarrow$ D accepts w

$w \notin L \Rightarrow M_2$ accept w
$\Rightarrow$ D rejects w

Classes of Languages



Reductions

A more useful undecidable language

$A_{\mathrm{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM that accepts input } w\}$

Theorem: A_{TM} is undecidable

Proof: Assume for the sake of contradiction that TM H decides A_{TM}:

$$H(\langle M, w\rangle) = \begin{cases} \text{accept} & \text{if } M \text{ accepts } w \\ \text{reject} & \text{if } M \text{ does not accept } w \end{cases}$$

Idea: Show that H can be used to decide the (undecidable) language SA_{TM} -- a contradiction.

"A reduction from SA_{TM} to A_{TM}"

Scientists vs. Engineers

A computer scientist and an engineer are stranded on a desert island. They find two palm trees with one coconut on each. The engineer climbs a tree, picks a coconut and eats.

The computer scientist climbs the second tree, picks a coconut, climbs down, climbs up the first tree and places it there, declaring success.

“Now we’ve reduced the problem to one we’ve already solved.”

Reductions

A reduction from problem A to problem B is an algorithm for problem A which uses an algorithm for problem B as a subroutine

If such a reduction exists, we say "A reduces to B"

Two uses of reductions

Algorithm for B ⇒ Algorithm for A

Positive uses: If A reduces to B and B is decidable, then A is also decidable

A

$EQ_{\text{DFA}} = \{\langle D_1, D_2\rangle \mid D_1, D_2 \text{ are DFAs and } L(D_1) = L(D_2)\}$

Theorem: EQ_{DFA} is decidable

Proof: The following TM decides EQ_{DFA}

On input $\langle D_1, D_2\rangle$, where $\langle D_1, D_2\rangle$ are DFAs:

1. Construct a DFA D that recognizes the symmetric difference $L(D_1) \triangle L(D_2)$

B

2. Run the decider for E_{DFA} on $\langle D\rangle$ and return its output

Two uses of reductions

Negative uses: If A reduces to B and A is undecidable, then B is also undecidable

B

$A_{\mathrm{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM that accepts input } w\}$

Suppose H decides A_{TM}

Consider the following TM D.

On input $\langle M\rangle$ where M is a TM:

1. Run H on input $\langle M, \langle M\rangle\rangle$
2. If H accepts, accept. If H rejects, reject.

Claim: D decides

$$SA_{\mathrm{TM}} = \{\langle M\rangle \mid M \text{ is a TM that accepts on input } \langle M\rangle\}$$

A

Two uses of reductions

Negative uses: If A reduces to B and A is undecidable, then B is also undecidable

Proof template:

1. Suppose to the contrary that B is decidable
2. Using B as a subroutine, construct an algorithm deciding A
3. But A is undecidable. Contradiction!

(conclude B not decidable)

Halting Problem

$$HALT_{\mathrm{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM that halts on input } w\}$$

Theorem: $HALT_{\mathrm{TM}}$ is undecidable

Proof: Suppose for contradiction that there exists a decider H for $HALT_{\mathrm{TM}}$. We construct a decider for A_{TM} as follows:

On input $\langle M, w\rangle$:

1. Run H on input $\langle M, w\rangle$
2. If H rejects, reject
3. If H accepts, simulate M on w
4. If M accepts, accept. Otherwise, reject

$\langle M,w\rangle \in A_{TM} \Rightarrow M$ halts and accepts on w
$\Rightarrow H(\langle M,w\rangle)$ accepts, M accepts
$\Rightarrow$ accepts

$\langle M,w\rangle \notin A_{TM}$:
case 1: M halts and rejects on w
$\Rightarrow H(\langle M,w\rangle)$ accepts, M rejects
$\Rightarrow$ reject
case 2: M loops on w
$\Rightarrow H(\langle M,w\rangle)$ rejects
$\Rightarrow$ reject

✗ because A_{TM} undecidable

This is a reduction from A_{TM} to $HALT_{\mathrm{TM}}$

Empty language testing for TMs

$$E_{\mathrm{TM}} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$$

Theorem: E_{TM} is undecidable

Proof: Suppose for contradiction that there exists a decider R for E_{TM}. We construct a decider for A_{TM} as follows:

On input $\langle M, w\rangle$:

1. Run R on input ???

This is a reduction from A_{TM} to E_{TM}

Empty language testing for TMs

$$E_{\mathrm{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$$

Theorem: E_{TM} is undecidable

Proof: Suppose for contradiction that there exists a decider R for E_{TM}. We construct a decider for A_{TM} as follows:

On input $\langle M, w \rangle$:

1. Construct a TM M' as follows:

   M' = " On input x :
   1) Ignore x
   2) Run M on w
   3) If M accepts, accept. If M rejects, reject. "

2. Run R on input $\langle M' \rangle$
3. If R rejects , accept. Otherwise, reject

Idea:

L(M') = ϕ if and only if M does not accept w

M accepts w ⇒ L(M') = Σ*

M does not accept w ⇒ L(M') = ϕ

This TM decides A_TM ✗

This is a reduction from A_{TM} to E_{TM}