BU CS 332 – Theory of Computation

Lecture 19:

- More on P
- Nondeterministic time, NP

HW 7 due on Monday

Reading:

Sipser Ch 7.2-7.3

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First topic: Time complexity

Last time: Answering the basic questions

1. How do we measure complexity? (as in CS 330)

2. Asymptotic notation (as in CS 330)

3. How robust is the TM model when we care about measuring complexity?

4. How do we mathematically capture our intuitive notion of “efficient algorithms”?

Running time and time complexity classes

input lengths

worst-case running times

Let $f : \mathbb{N} \to \mathbb{N}$

$f(n) = \log n$ really means $f(n) = \lceil \log n \rceil$

A TM M runs in time $f(n)$ if on every input $w \in \Sigma^n$,

M halts on w within at most $f(n)$ steps

$\mathrm{TIME}(f(n))$ is a class (i.e., set) of languages:

A language $A \in \mathrm{TIME}(f(n))$ if there exists a basic single-tape (deterministic) TM M that

1) Decides A, and

2) Runs in time $O(f(n))$

Complexity Class $\mathbf{P}$

Complexity class P

Definition: P is the class of languages decidable in polynomial time on a basic single-tape (deterministic) TM

$$\mathrm{P} = \bigcup_{k=1}^{\infty} \mathrm{TIME}(n^k)$$

$SUM = \{\langle x, y, z\rangle \mid x + y = z\} \in P$

- Class doesn't change if we substitute in another reasonable deterministic model (Extended Church-Turing)
- Cobham-Edmonds Thesis: Captures class of problems that are feasible to solve on computers

A note about encodings

We'll still use the notation $\langle \quad \rangle$ for "any reasonable" encoding of the input to a TM…but now we have to be more careful about what we mean by "reasonable"

How long is the encoding of a k-vertex graph…

… as an adjacency matrix? $O(k^2)$

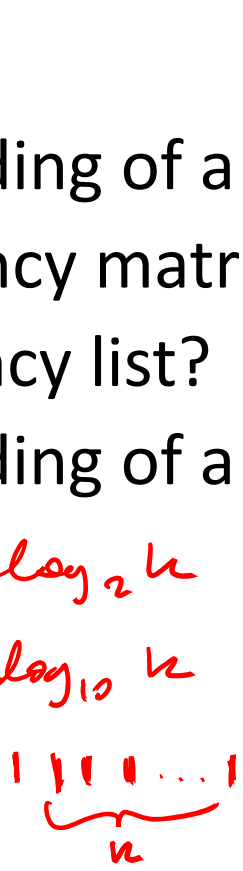
… as an adjacency list?

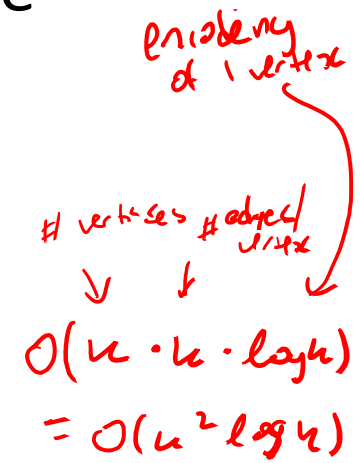
How long is the encoding of a natural number k

… in binary? $\log_2 k = O(\log k)$

… in decimal? $\log_{10} k = O(\log k)$

… in unary? $O(k)$





An alg. running in time $O(k)$ would be an

- $O(k)$-time alg. in unary encoding
- $2^{O(n)}$-time alg. in binary encoding

Describing and analyzing polynomial-time algorithms

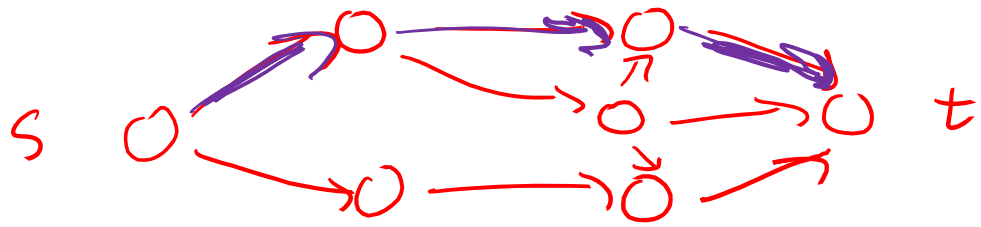
- Due to Extended Church-Turing Thesis, we can still use high-level descriptions on multi-tape machines
- Polynomial-time is robust under composition: $\mathrm{poly}(n)$ executions of $\mathrm{poly}(n)$-time subroutines run on $\mathrm{poly}(n)$-size inputs gives an algorithm running in $\mathrm{poly}(n)$ time.

  => Can freely use algorithms we've seen before as subroutines if we've analyzed their runtime

- Need to be careful about size of inputs! (Assume inputs represented in binary unless otherwise stated.)

Examples of languages in P

- $PATH = \{\langle G, s, t\rangle \mid G \text{ is a directed graph with a directed path from } s \text{ to } t\}$



Breadth-first search
(Runs in time $O(\#\text{edges})$)

- $E_{\text{DFA}} = \{\langle D\rangle \mid D \text{ is a DFA that recognizes the empty language}\}$

To decide $\overline{E_{\text{DFA}}}$, suffices to determine if an accept state is reachable from start state (Breadth-first search)

Examples of languages in P

- $RELPRIME = \{\langle x, y \rangle \mid x \text{ and } y \text{ are relatively prime}\}$

means $\gcd(x, y) = 1$

Euclid's Algorithm runs in polynomial time in binary encoding of x, y

- $PRIMES = \{\langle x \rangle \mid x \text{ is prime}\}$

2006 Gödel Prize citation



The 2006 Gödel Prize for outstanding articles in theoretical computer science is awarded to Manindra Agrawal, Neeraj Kayal, and Nitin Saxena for their paper "PRIMES is in P."

In August 2002 one of the most ancient computational problems was finally solved….

A polynomial-time algorithm for $PRIMES$?

Consider the following algorithm for $PRIMES$

On input $\langle x \rangle$:

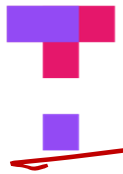
For $b = 2, 3, 5, \ldots, \sqrt{x}$:

Try to divide x by b

If b divides x, [redacted] reject

If all b fail to divide x, [redacted] accept

How many divisions does this algorithm require in terms of $n = |\langle x \rangle|$?



CFG Generation

Theorem: Every context-free language is in P

Chomsky Normal Form for CFGs:

- Can have a rule $S \to \varepsilon$
- All remaining rules of the form $A \to BC$ or $A \to a$
- Cannot have S on the RHS of any rule

Lemma: Any CFG can be converted into an equivalent CFG in Chomsky Normal Form

Lemma: If G is in Chomsky Normal Form, any nonempty string w that can be derived from G has a derivation with at most $2|w| - 1$ steps

CFG Generation

Theorem: Every context-free language is in P

Proof attempt: Let A be a CFL with a CFG G in Chomsky Normal Form. Define a TM M recognizing A:

On input w: (Let $n = |w|$)

1. Enumerate all strings derivable in $\leq 2n - 1$ steps
2. If any of these strings equal w, accept. Else, reject.

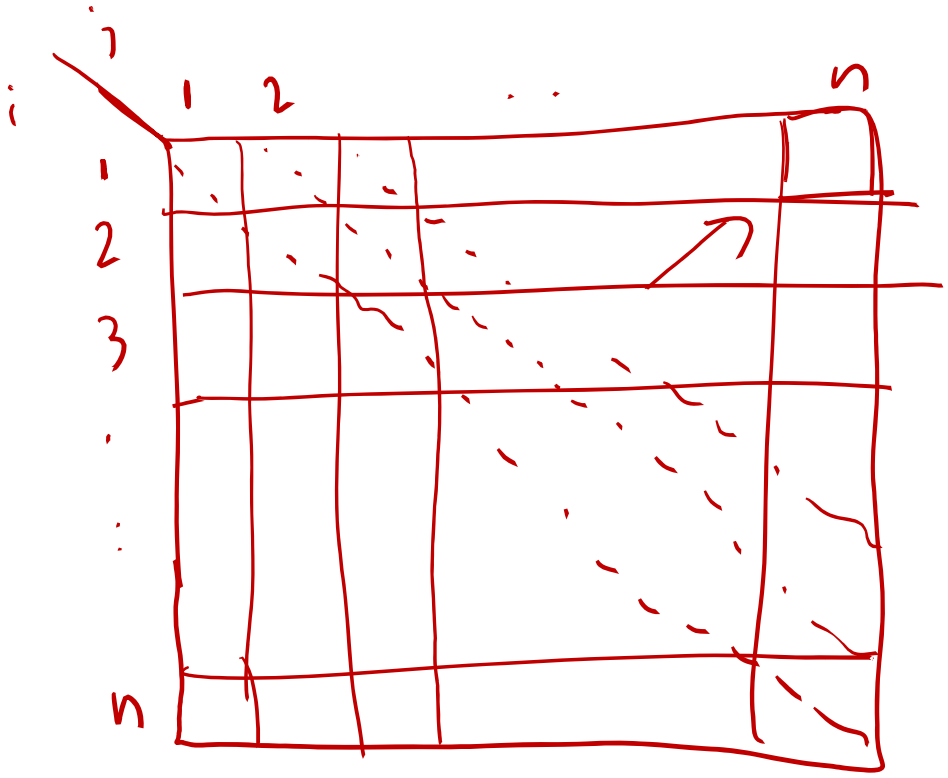
What is the running time of this algorithm? Exponential in n

A better idea: Use dynamic programming

- Identify subproblems
- Record solutions to subproblems in a table
- Solve bigger subproblems by combining solutions to smaller subproblems

Subproblem: "Does variable A generate substring $w_i w_{i+1} \dots w_j$ of w?"

Original problem: $A: S \quad i = 1, \; j = n$



Place variable A in cell (i,j)

$\Leftrightarrow$

A generates $w_i \dots w_j$

Total time: $O(n^3)$

Running time:

Outer loop: Loop over n substring lengths

Blue loop: Loop over all $O(n)$ cuts

Purple loop: Loop over $O(n)$ substrings

Green loop: Loop over all rules and check $O(1)$

(1) Solve problems for substrings of length 1

For each variable $A \in V$:

If $A \to w_i$ is a rule, add A to cell (i,i)

(2) Solve problems for substrings of length 2

For each substring $w_i w_{i+1}$:

If $A \to BC$ is a rule

and $B \in$ cell (i,i)

and $C \in$ cell $(i+1, i+1)$

add A to cell $(i, i+1)$

$V = w_{i+1} \dots w_{i+k-1}$

(k) Solve problems for substrings of length k

→ For each $w_i w_{i+1} \dots w_{i+k-1}$

→ For each way of putting $w_i \dots w_{i+k-1}$ into 2 substrings $u = w_i \dots w_{i+l-1}$, $v = w_{i+l} \dots w_{i+k-1}$

→ If $A \to BC$ a rule and $B \in$ cell $(i, i+l)$ and $C \in$ cell $(i+l+1, i+k-1)$

add A to cell $(i, i+k-1)$

Examples of languages in P

Problem	Description	Algorithm	Yes	No
MULTIPLE	Is x a multiple of y?	Grade school division	51, 17	51, 16
RELPRIME	Are x and y relatively prime?	Euclid (300 BCE)	34, 39	34, 51
PRIMES	Is x prime?	AKS (2002)	53	51
all CFLs (e.g. the language of balanced parentheses and brackets)	Is a given string in a fixed CFL? (E.g., is the string of parentheses and brackets balanced?)	Dynamic programming	Depends on the language; e.g. (([])[])	Depends on the language; e.g. ([)], (()
LSOLVE	Is there a vector x that satisfies Ax = b?	Gauss-Edmonds elimination	$\begin{bmatrix}0 & 1 & 1\\ 2 & 4 & -2\\ 0 & 3 & 15\end{bmatrix}, \begin{bmatrix}2\\4\\36\end{bmatrix}$	$\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 1\\ 0 & 1 & 1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$

Beyond polynomial time

Definition: EXP is the class of languages decidable in exponential time on a basic single-tape (deterministic) TM

$$\mathrm{EXP} = \bigcup_{k=1}^{\infty} \mathrm{TIME}(2^{n^k})$$

Nondeterministic Time and NP

Nondeterministic time

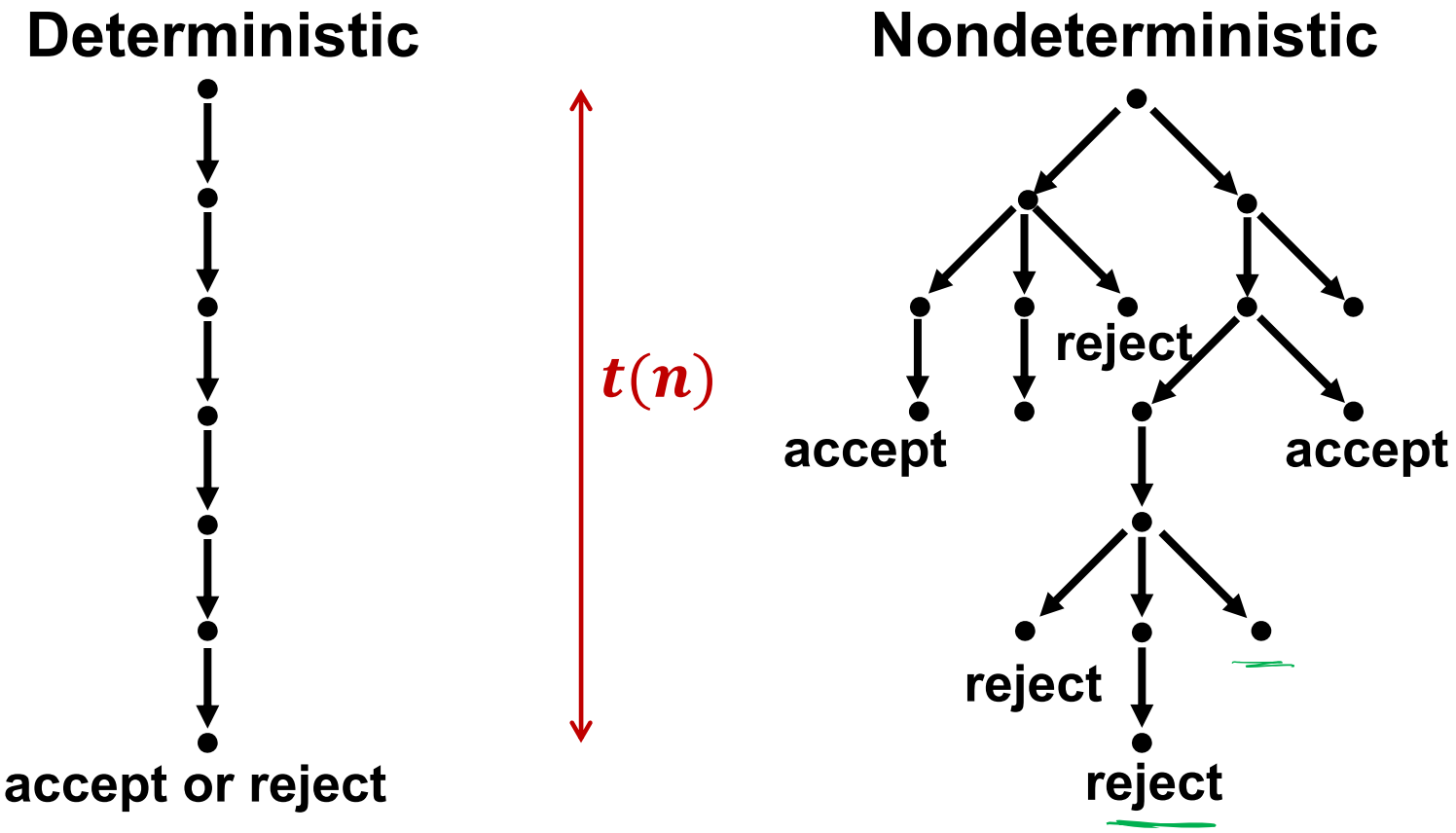
Let $f : \mathbb{N} \rightarrow \mathbb{N}$

A NTM M runs in time $f(n)$ if on every input $w \in \Sigma^n$,

M halts on w within at most $f(n)$ steps on every computational branch

"or dies w/ no transitions"

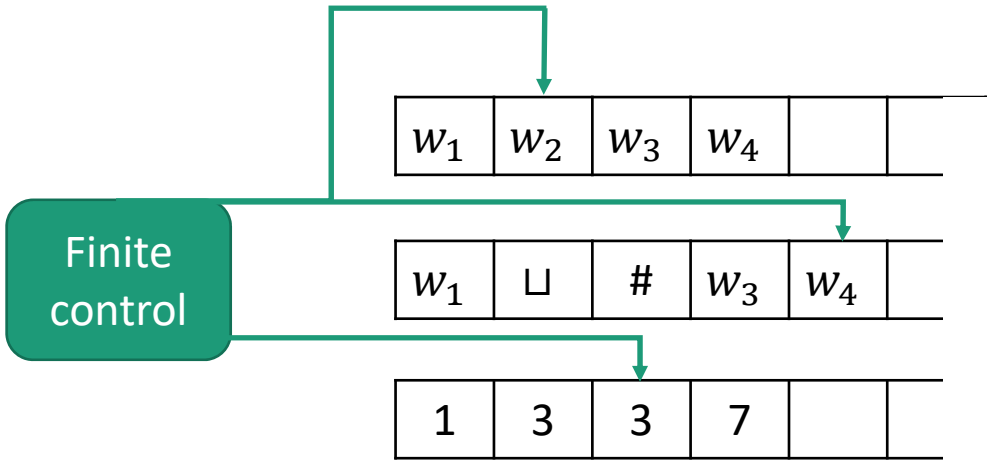
Deterministic vs. nondeterministic time



Deterministic vs. nondeterministic time

Theorem: Let $t(n) \geq n$ be a function. Every NTM running in time $t(n)$ has an equivalent single-tape TM running in time $2^{O(t(n))}$

Proof: Simulate NTM by 3-tape TM



Input w to N (read-only)

Simulation tape (run N on w using nondeterministic choices from tape 3)

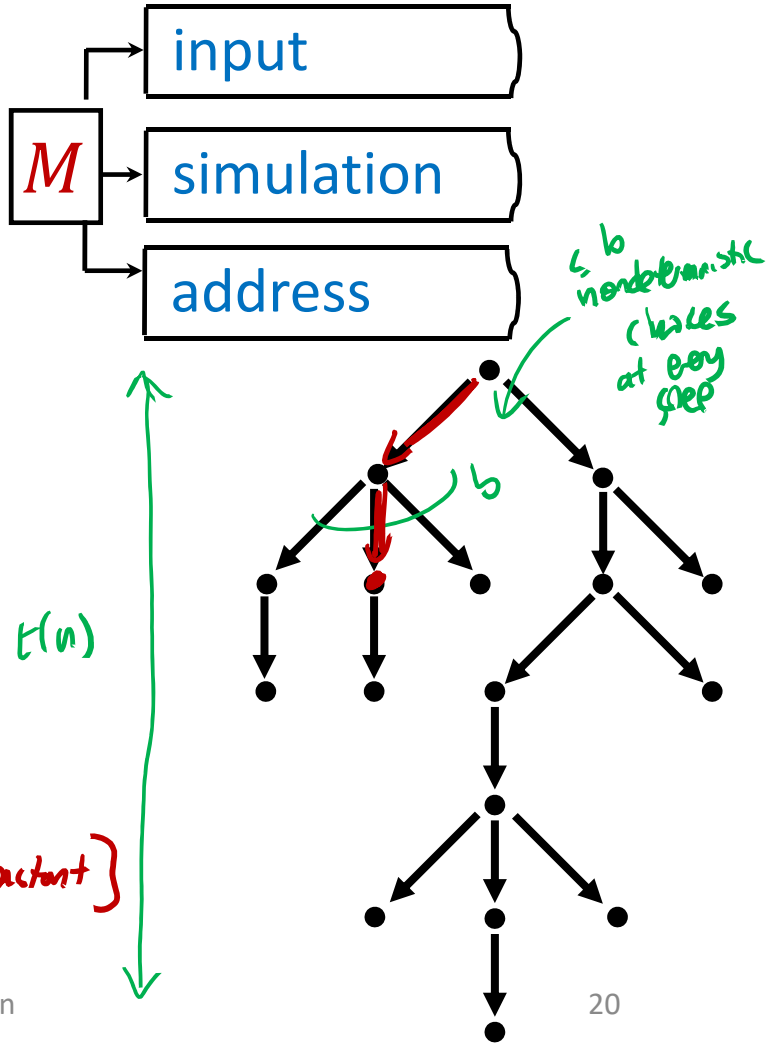
Address in computation tree

Deterministic vs. nondeterministic time

Theorem: Let $t(n) \geq n$ be a function. Every NTM running in time $t(n)$ has an equivalent single-tape TM running in time $2^{O(t(n))}$

Proof: Simulate NTM by 3-tape TM

- # leaves: $b^{t(n)}$
- # nodes: $2 \cdot b^{t(n)}$



Running time:

To simulate one root-to-node path: $O(t(n))$

Total time: $O(2 \cdot b^{t(n)} \cdot t(n)) = 2^{O(t(n))}$ [b constant]

Deterministic vs. nondeterministic time

Theorem: Let $t(n) \geq n$ be a function. Every NTM running in time $t(n)$ has an equivalent single-tape TM running in time $2^{O(t(n))}$

Proof: Simulate NTM by 3-tape TM in time $2^{O(t(n))}$

We know that a 3-tape TM can be simulated by a single-tape TM with quadratic overhead, hence we get running time

$$(2^{O(t(n))})^2 = 2^{2 \cdot O(t(n))} = 2^{O(t(n))}$$