# BU CS 332 – Theory of Computation

## Lecture 4:

- More on NFAs
- NFAs vs. DFAs
- Closure Properties

Reading:

Sipser Ch 1.1-1.2

HW 1 Thursday 11:59pm

Gradescope checkins

Mark Bun

February 3, 2021

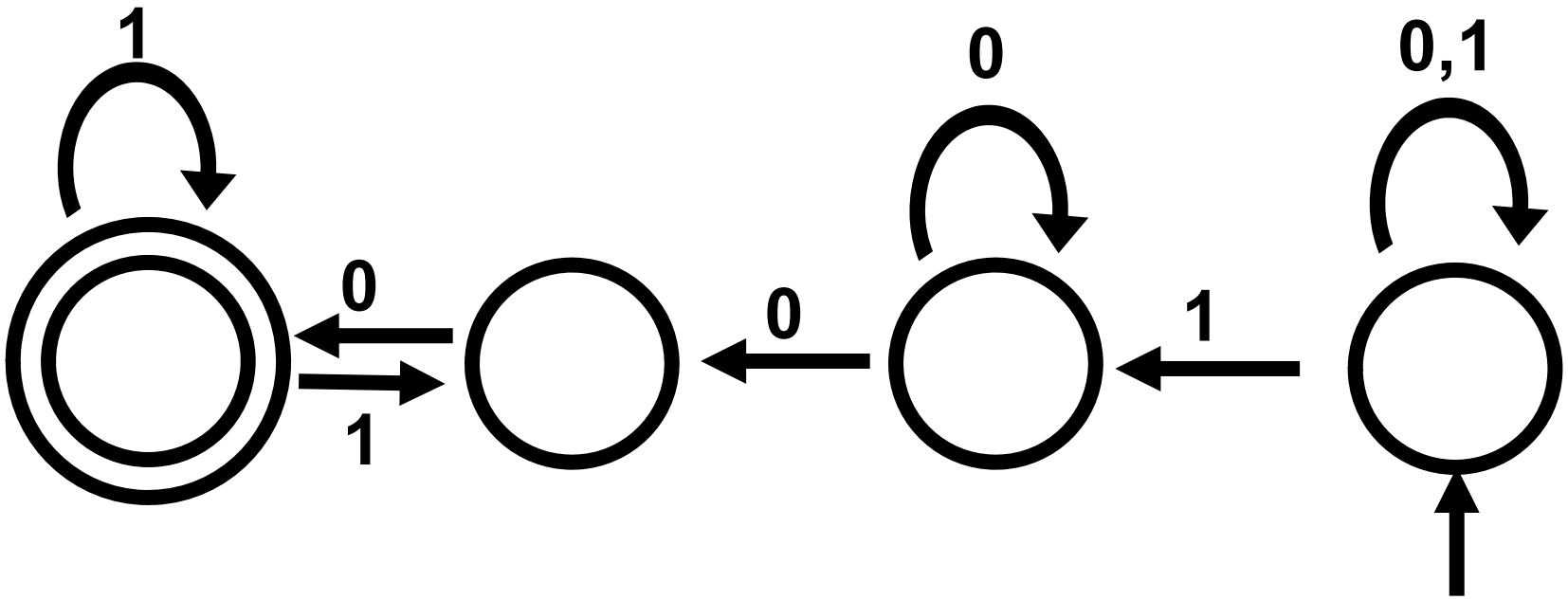
# Last Time

- Deterministic Finite Automata (DFAs)
  - Informal description: State diagram
  - Formal description: What are they?
  - Formal description: How do they compute?

  - A language is regular if it is recognized by a DFA

- Intro to Nondeterministic FAs

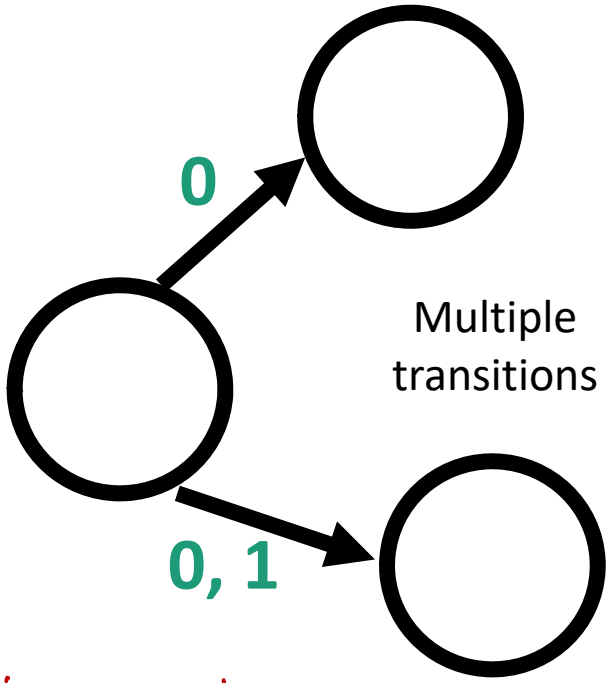
# Nondeterminism




**A Nondeterministic Finite Automaton (NFA) accepts if there *exists* a way to make it reach an accept state.**

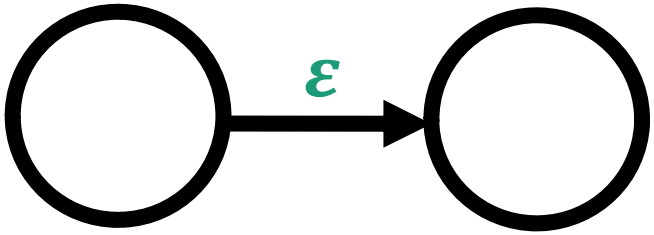
An NFA fails to accept an input if every computation path fails to lead to accept state

# Some special transitions




Multiple transitions

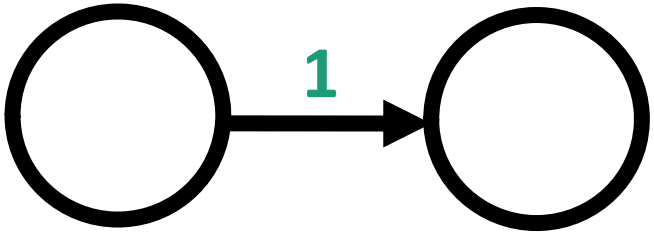
$\varepsilon$-transitions
(don't consume a symbol)



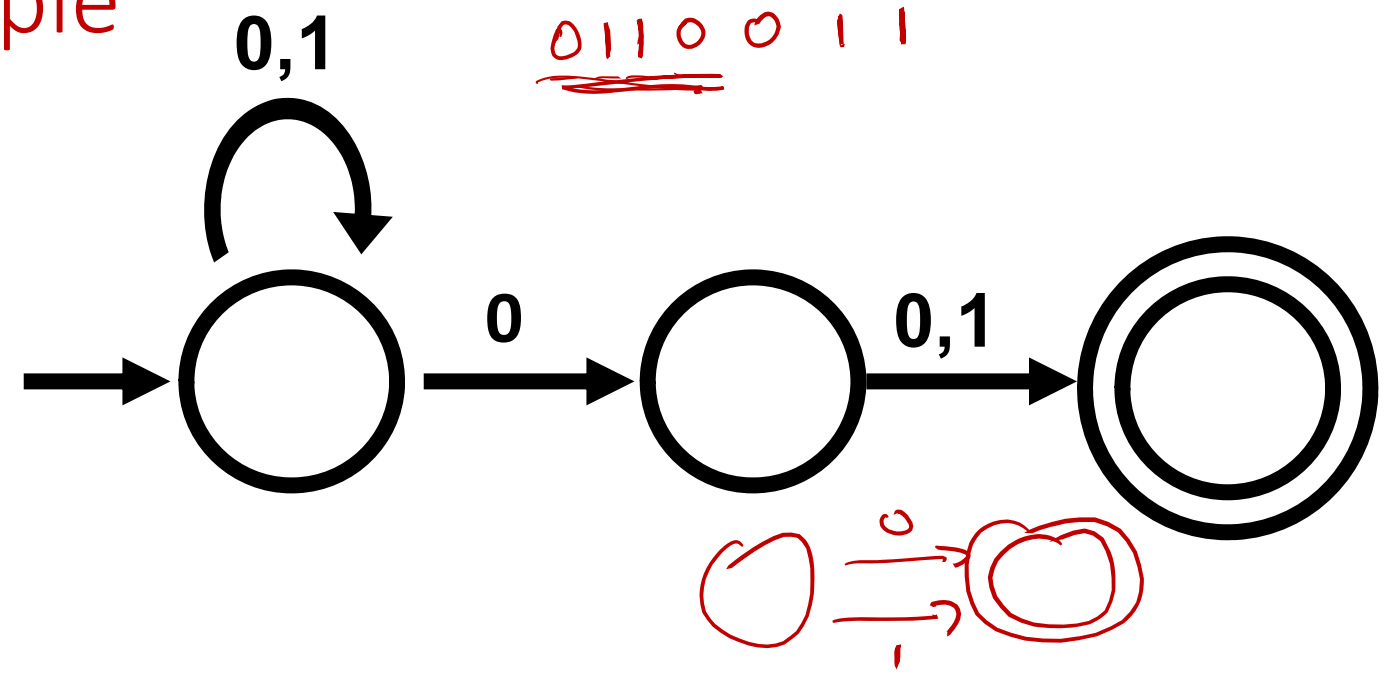

DFA: Can't have these kinds of transitions

Exactly 1 outgoing trans. per alphabet symbol

No transition



# Example




$\boldsymbol{L(N)} =$

a) $\{w \mid w$ contains 00 or 01$\}$

b) $\{w \mid$ the second to last symbol of $w$ is 0$\}$

c) $\{w \mid w$ starts with 00 or 01$\}$

d) $\{w \mid w$ ends with 001$\}$

# Formal Definition of a NFA

An NFA is a 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$

$Q$ is the set of states

$\Sigma$ is the alphabet

$\delta$: $Q \times \Sigma_\varepsilon \rightarrow P(Q)$ is the transition function

$q_0 \in Q$ is the start state

$F \subseteq Q$ is the set of accept states

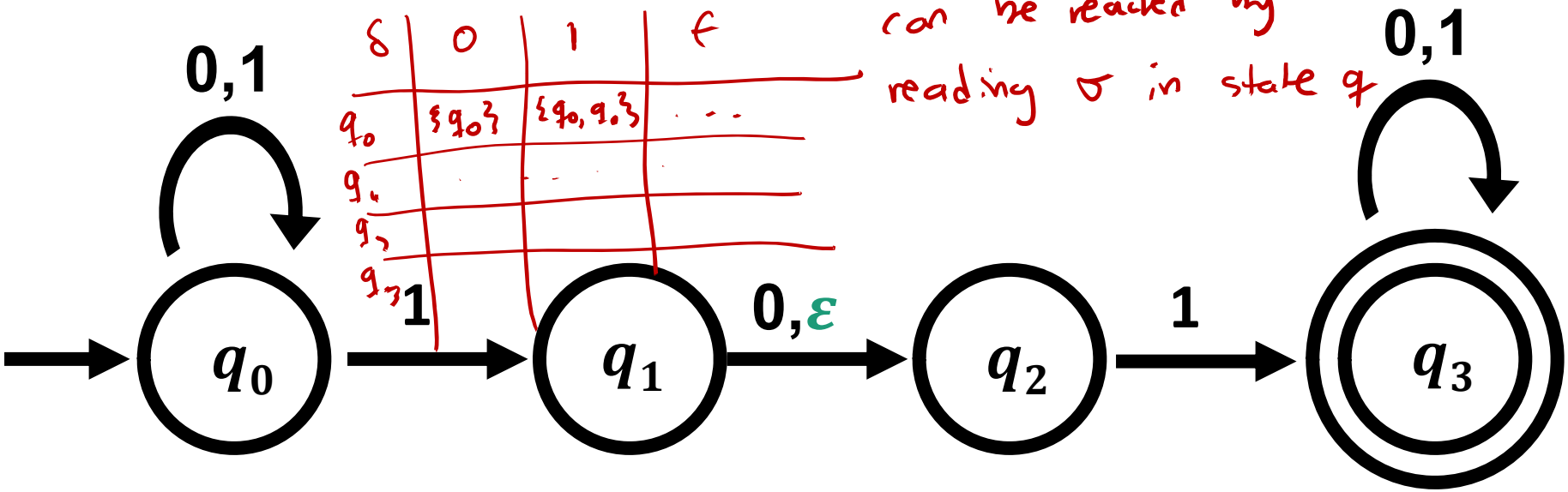
$\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$

$P(Q)$ = power set of $Q$ = $\{R \mid R \subseteq Q\}$

$M$ ***accepts*** a string $w$ if **there exists** a path from $q_0$ to an accept state that can be followed by reading $w$.

# Example

$\delta(q, \sigma)$ = set of states that can be reached by reading $\sigma$ in state $q$




| δ | 0 | 1 | ε |
|---|---|---|---|
| $q_0$ | $\{q_0\}$ | $\{q_0, q_1\}$ | ... |
| $q_1$ | | | |
| $q_2$ | | | |
| $q_3$ | | | |

$N = (Q, \Sigma, \delta, q_0, F)$

$Q = \{q_0, q_1, q_2, q_3\}$

$\Sigma = \{0, 1\}$ $\quad \Sigma_\epsilon : \{\epsilon, 0, 1\}$

$F = \{q_3\}$

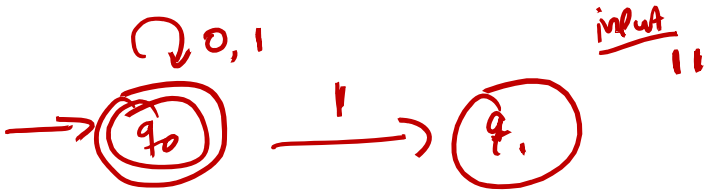
$\delta(q_0, 0) = \{q_0\}$

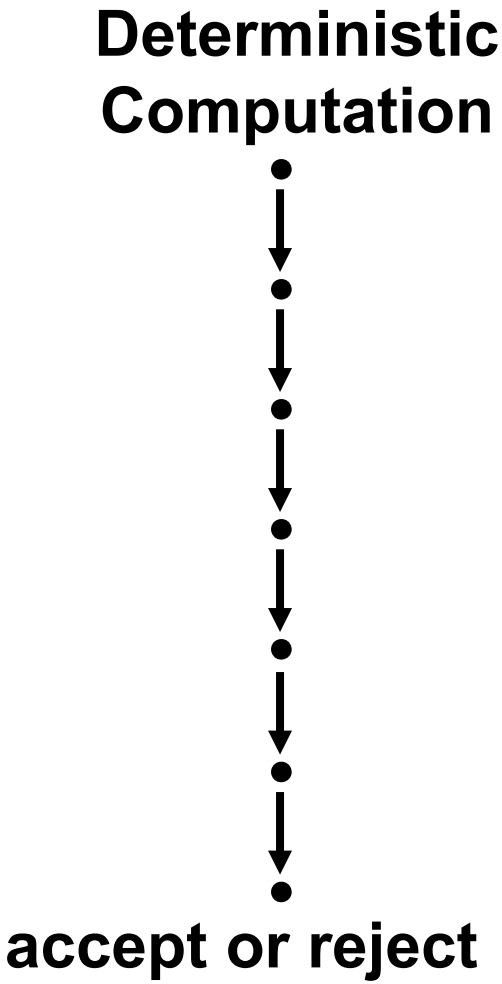
$\delta(q_0, 1) = \{q_0, q_1\}$

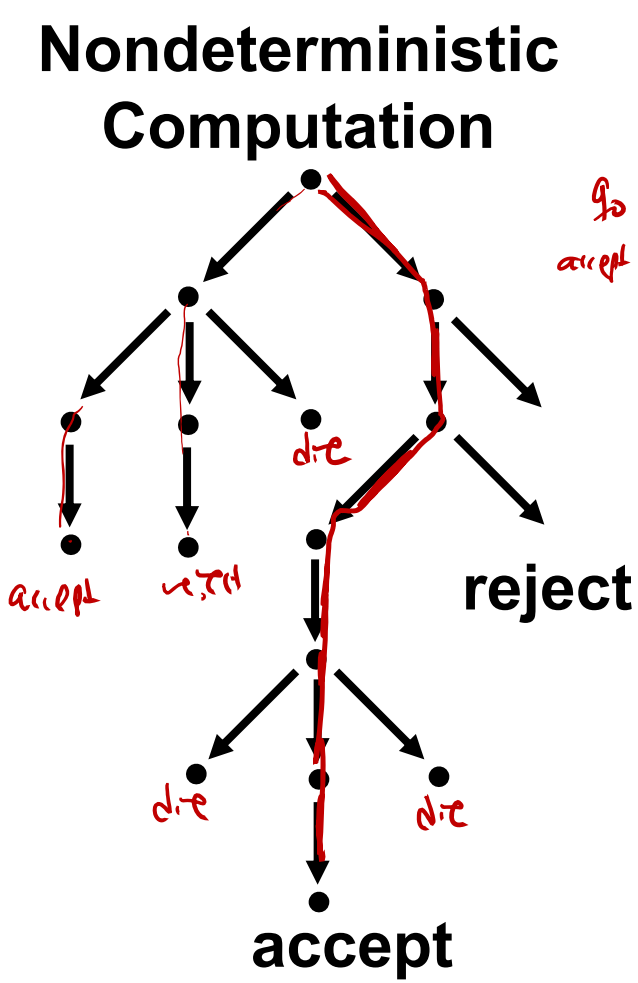
$\delta(q_1, \varepsilon) = \{q_2\}$

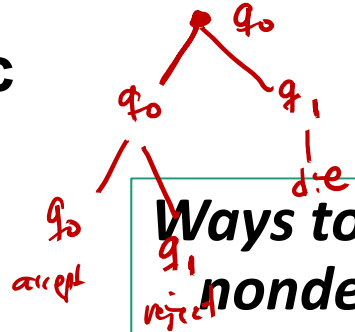
$\delta(q_2, 0) = \phi$

# Nondeterminism




***Ways to think about nondeterminism***

- **(restricted) parallel computation**
- **tree of possible computations**
- **guessing and verifying the "right" choice**

# Why study NFAs?

- Not really a realistic model of computation: Real computing devices can't really try many possibilities in parallel

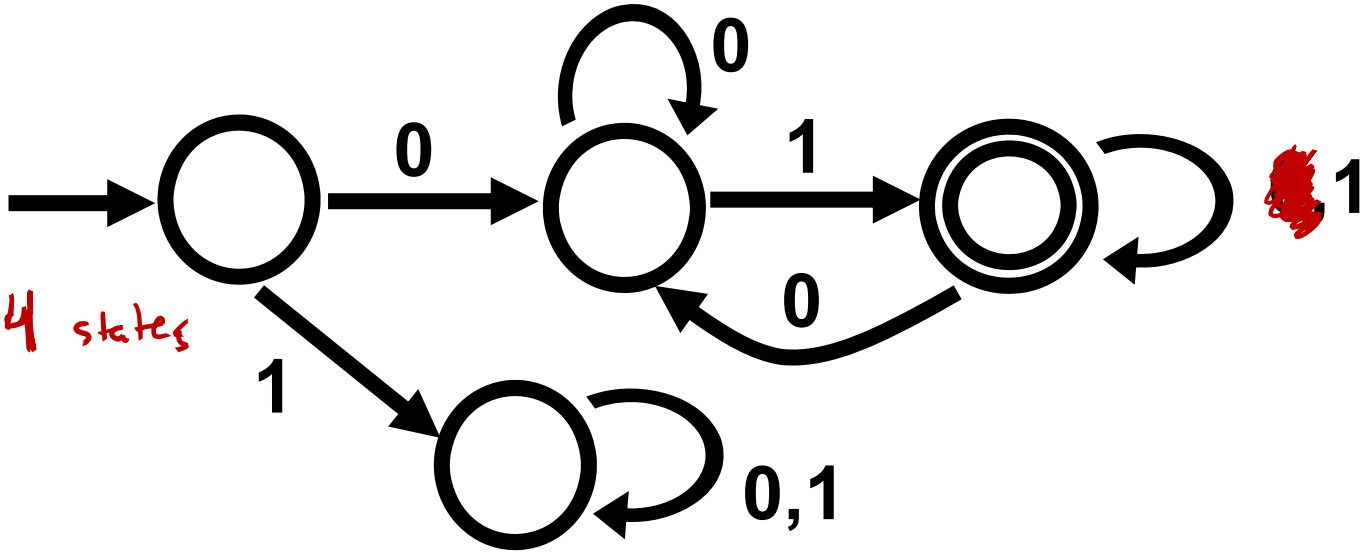
But:

- Useful tool for understanding power of DFAs/regular languages
- NFAs can be simpler than DFAs
- Lets us study "nondeterminism" as a resource

  (cf. P vs. NP)

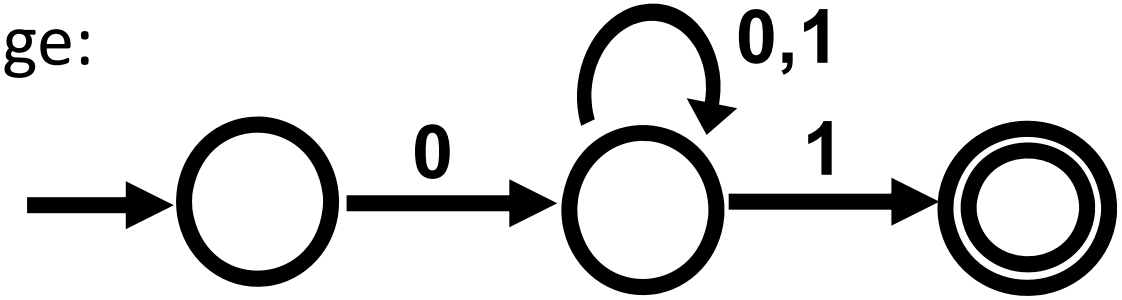
# NFAs can be simpler than DFAs

A DFA that recognizes the language
$\{w \mid w \text{ starts with } 0 \text{ and ends with } 1\}$:




Any DFA needs 4 states

An NFA for this language:



# Equivalence of NFAs and DFAs

# Equivalence of NFAs and DFAs

Every DFA *is* an NFA, so NFAs are *at least* as powerful as DFAs

regular languages $\subseteq$ languages recognizable by NFAs

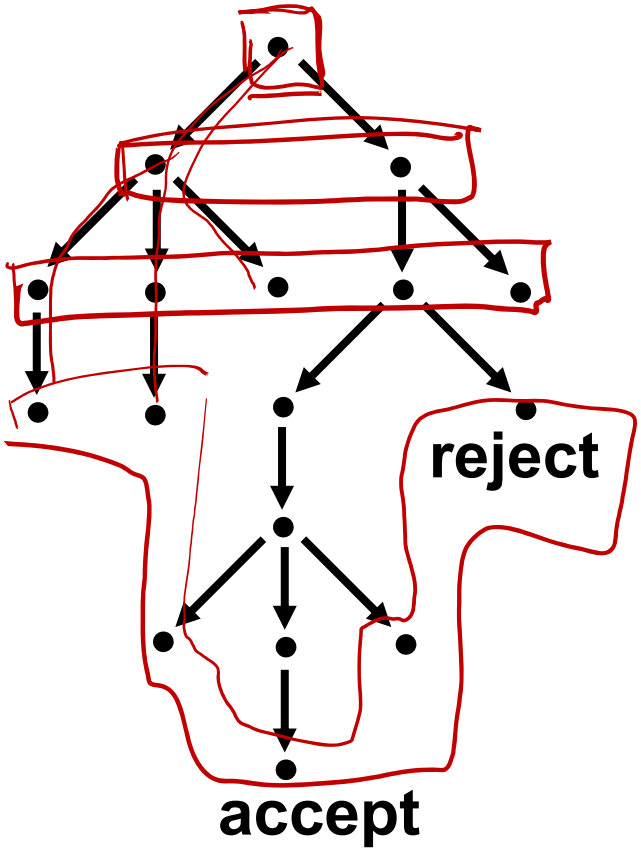
Theorem: For every NFA $N$, there is a DFA $M$ such that $L(M) = L(N)$ $\supseteq$

Corollary: A language is regular if and only if it is recognized by an NFA

# Equivalence of NFAs and DFAs (Proof)

Let $N = (Q, \Sigma, \delta, q_0, F)$ be an NFA

Goal: Construct DFA $M = (Q', \Sigma, \delta', q_0', F')$ recognizing $L(N)$

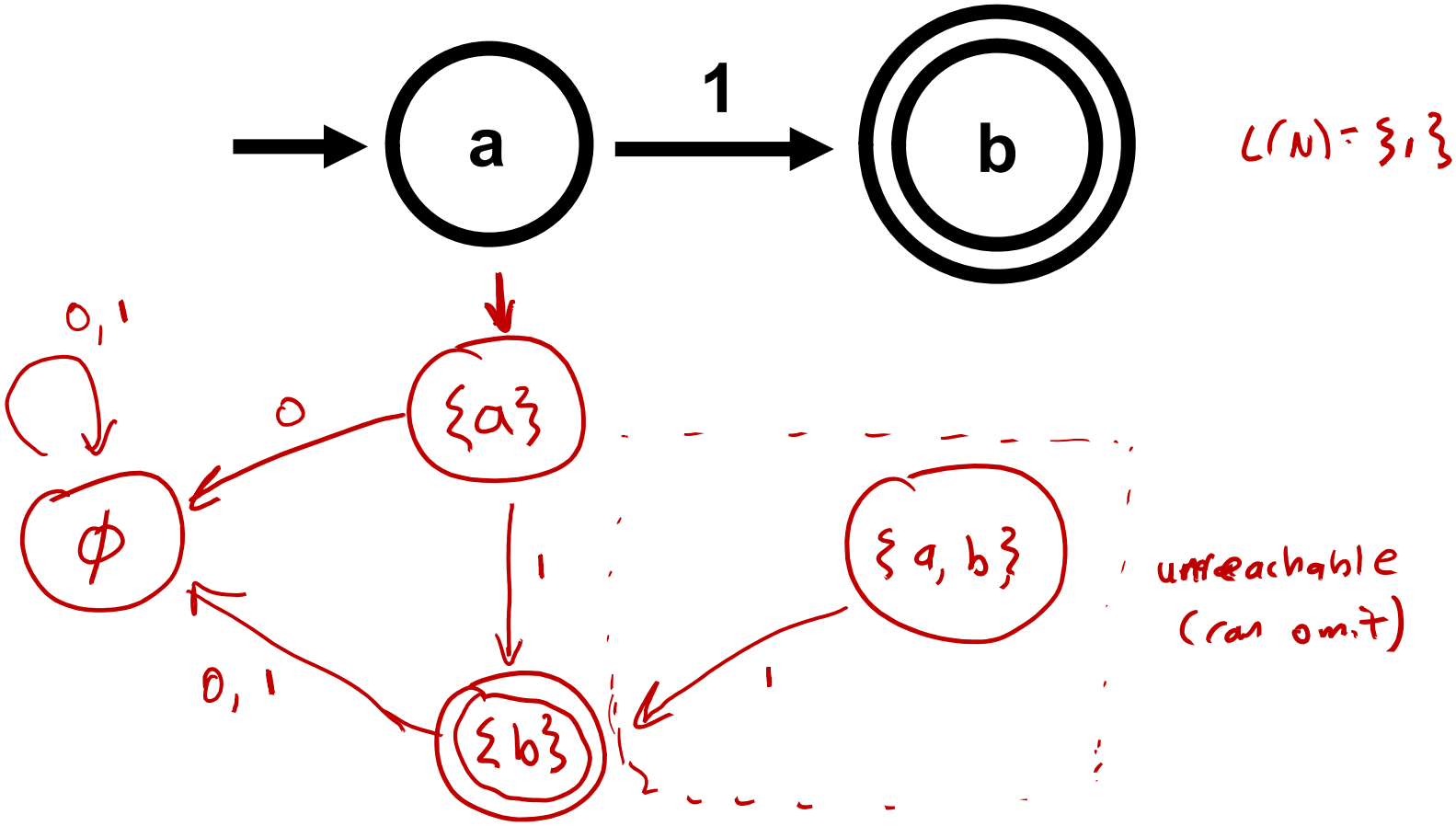



Intuition: Run all threads of $N$ in parallel, maintaining the set of states where all threads are.

Formally: $Q' = P(Q)$

"The Subset Construction"

# NFA -> DFA Example (over {0,1})



# Subset Construction (Formally, first attempt)

Input: NFA $N = (Q, \Sigma, \delta, q_0, F)$

Output: DFA $M = (Q', \Sigma, \delta', q_0', F')$

$Q' = P(Q)$ { set of all subsets of Q }

$\delta' : Q' \times \Sigma \rightarrow Q'$

$\delta'(R, \sigma) = \bigcup_{r \in R} \delta(r, \sigma)$ for all $R \subseteq Q$ and $\sigma \in \Sigma$.

$q_0' = \{q_0\}$

$F' = \{ R \in Q' \mid R$ contains some $q \in F \}$

# Subset Construction (Formally, for real)

Input: NFA $N = (Q, \Sigma, \delta, q_0, F)$

Output: DFA $M = (Q', \Sigma, \delta', q_0', F')$

For $R \subseteq Q$, define
$E(R)$ = { states reachable from $R$ using $\in$ transitions }

$Q' = P(Q)$

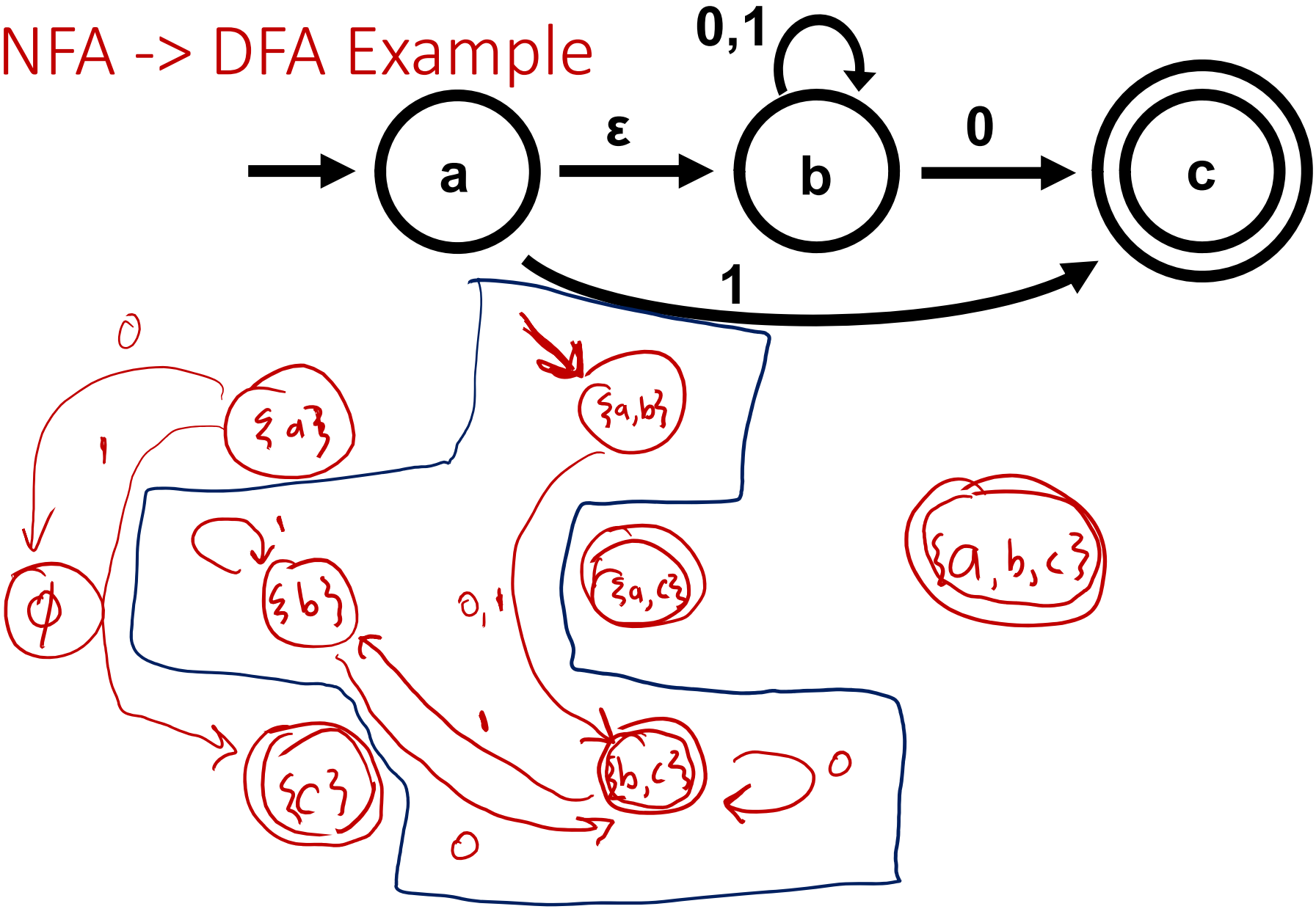
$\delta' : Q' \times \Sigma \rightarrow Q'$

$\delta'(R, \sigma) = \bigcup_{r \in R} E(\delta(r, \sigma))$ for all $R \subseteq Q$ and $\sigma \in \Sigma$.

$q_0' = E(\{q_0\})$

$F' = \{ R \in Q' \mid R$ contains some accept state of $N \}$

# NFA -> DFA Example



# Proving the Construction Works

**Claim:** For every string $w$, running $M$ on $w$ leads to state

$$\{q \in Q | \text{There exists a computation path of } N \text{ on input } w \text{ ending at } q\}$$

**Proof idea:** By induction on $|w|$

# Historical Note

Subset Construction introduced in Rabin & Scott's 1959 paper "Finite Automata and their Decision Problems"

1976 ACM Turing Award citation

For their joint paper "Finite Automata and Their Decision Problem," which introduced the idea of nondeterministic machines, which has proved to be an enormously valuable concept. Their (Scott & Rabin) classic paper has been a continuous source of inspiration for subsequent work in this field.

# NFA -> DFA: The Catch

If $N$ is an NFA with $s$ states, how many states does the DFA obtained using the subset construction have?

a) $s$

b) $s^2$

c) $2^s$

d) None of the above

# DFA $= |P(Q)|$

$Q$: state set of NFA

$|P(Q)| = 2^{|Q|}$

# Is this construction the best we can do?

Subset construction converts an $n$ state NFA into a $2^n$-state DFA

Could there be a construction that always produces, say, an $n^2$-state DFA?

Theorem: For every $n \geq 1$, there is a language $L_n$ such that

1. There is an $(n+1)$-state NFA recognizing $L_n$.
2. There is no DFA recognizing $L_n$ with fewer than $2^n$ states.

Conclusion: For finite automata, nondeterminism provides an exponential savings over determinism (in the worst case).

# Closure Properties

# An Analogy

In algebra, we try to identify operations which are common to many different mathematical structures

Example: The integers $\mathbb{Z} = \{\ldots -2, -1, 0, 1, 2, \ldots\}$ are **closed** under

- Addition: $x + y$
- Multiplication: $x \times y$
- Negation: $-x$
- …but NOT Division: $x \,/\, y$

We'd like to investigate similar closure properties of the class of regular languages

# Regular operations on languages

Let $A, B \subseteq \Sigma^*$ be languages. Define

Union: $A \cup B = \{w \mid w \in A \textbf{ or } w \in B\}$

Concatenation: $A \circ B = \{xy \mid x \in A, y \in B\}$

Star: $A^* = \{a_1 a_2 \ldots a_n \mid a_i \in A, n \geq 0\}$

each $a_i$ is a string, could be different

$= \{\epsilon\} \cup A \cup A \circ A \cup A \circ A \circ A \cup \ldots$

# Other operations

Let $A, B \subseteq \Sigma^*$ be languages. Define

Complement: $\bar{A} = \{w \mid w \notin A\}$

Intersection: $A \cap B = \{w \mid w \in A \textbf{ and } w \in B\}$

Reverse: $A^R = \{w \mid w^R \in A\}$

# Closure properties of the regular languages

**Theorem:** The class of regular languages is closed under all three regular operations (union, concatenation, star), as well as under complement, intersection, and reverse.

i.e., if $A$ and $B$ are regular, applying any of these operations yields a regular language