BU CS 332 – Theory of Computation

Lecture 11:

- TM Variants and Closure Properties
- Church-Turing Thesis

Reading:

Sipser Ch 3.2

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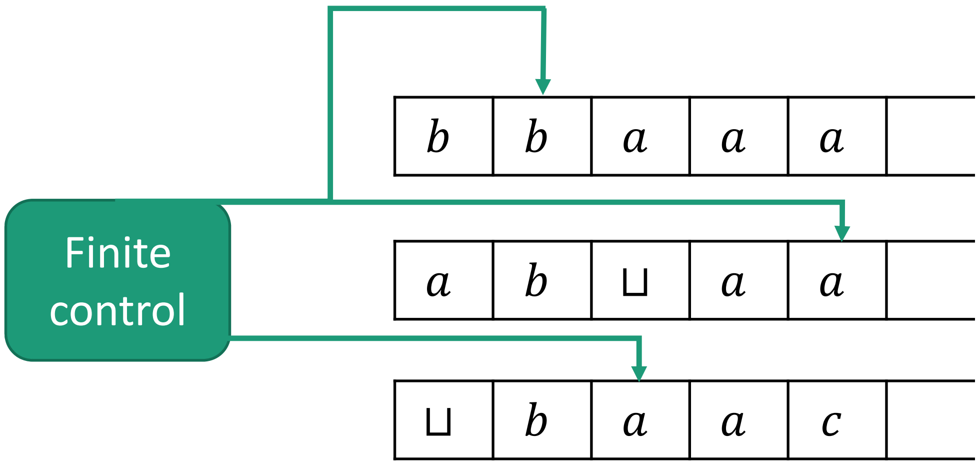
TM Variants

TMs are equivalent to…

- TMs with “stay put”
- TMs with 2-way infinite tapes
- Multi-tape TMs
- Nondeterministic TMs
- Random access TMs
- Enumerators
- Finite automata with access to an unbounded queue
- Primitive recursive functions
- Cellular automata

…

Multi-Tape TMs



Input
(read only)

work tapes
(read & write)

independent of input

Fixed number of tapes k (can't change during computation)

Transition function $\delta : Q \times \Gamma^k \rightarrow Q \times \Gamma^k \times \{L, R, S\}^k$

current state | k reads | next state | k writes | k movements

How to Simulate It

(multi-tape)
To show that a TM variant is no more powerful than the basic, single-tape TM:

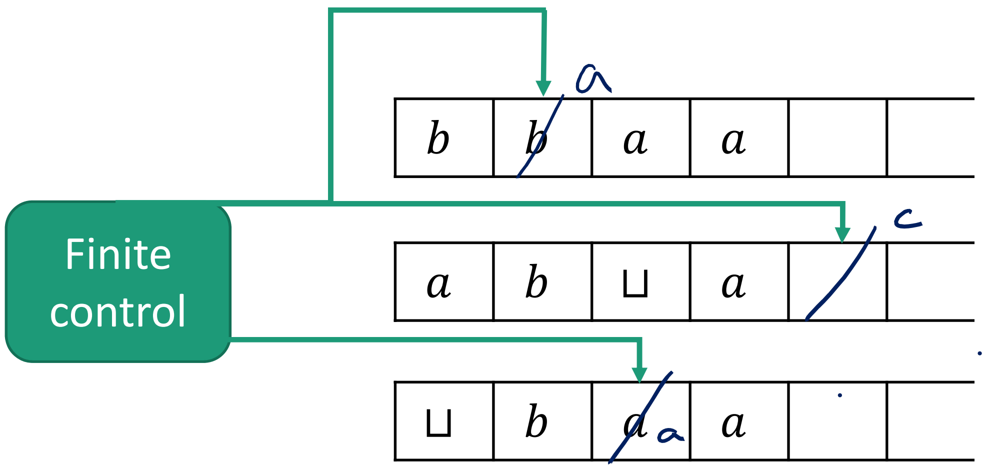
Show that if M is any variant machine, there exists a basic, single-tape TM M' that can simulate M

(Usual) parts of the simulation:

- Describe how to initialize the tapes of M' based on the input to M
- Describe how to simulate one step of M's computation using (possibly many steps of) M'

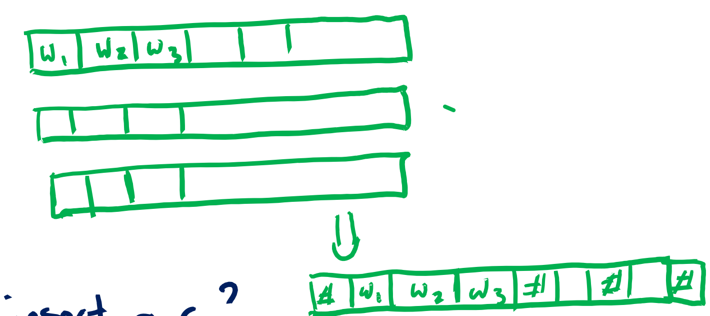
Multi-Tape TMs are Equivalent to Single-Tape TMs

Theorem: Every k-tape TM M with can be simulated by an equivalent single-tape TM M'

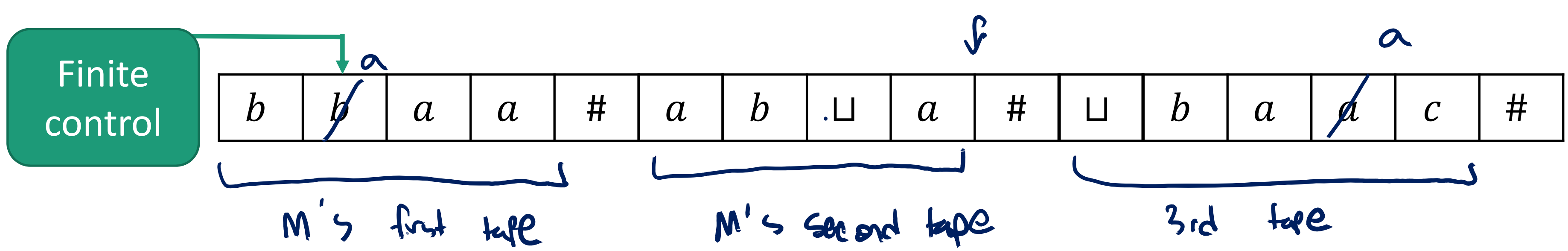


1) Initialize M' tape

2) Simulate each step of M on M'



insert a c?



Simulating Multiple Tapes

tape alphabet of M' $= \Gamma \cup \{\dot{a} \mid a \in \Gamma\}$

Implementation-Level Description of M'

$\dot{a}$ means tape head is over this symbol

remaining simulated tapes blank

input on first simulated tape

On input $w = w_1 w_2 \ldots w_n$

1. Format tape into $\# \, \dot{w}_1 w_2 \ldots w_n \# \, \dot{\sqcup} \, \# \, \dot{\sqcup} \, \# \ldots \#$
2. For each move of M:

   Scan left-to-right, finding current symbols

   Scan left-to-right, writing new symbols,

   Scan left-to-right, moving each tape head

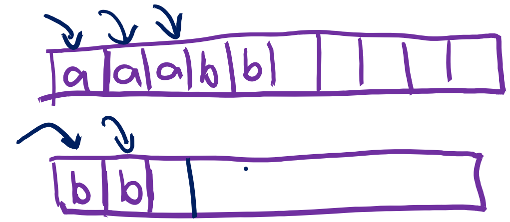
   If a tape head goes off the right end, insert blank

   If a tape head goes off left end, move back right

Why are Multi-Tape TMs Helpful?

To show a language is Turing-recognizable or decidable, it's enough to construct a multi-tape TM

Often easier to construct multi-tape TMs



Ex. Decider for $\{a^i b^j \mid i > j\}$

On input w:

1) Check w/ a left-right scan $w \in L(a^* b^*)$
2) Copy all b's from w to tape 2
3) Starting from left ends of tapes 1 and 2, check that every b on tape 2 has an accompanying a on tape 1
4) Check first blank on tape 2 has an accompanying a on tape 1

Why are Multi-Tape TMs Helpful?

To show a language is Turing-recognizable or decidable, it's enough to construct a multi-tape TM

1) M_1 accepts w $(w \in L_1)$
$\Rightarrow M$ accepts w ✓

2) M_1 rejects w, M_2 accepts w
$\Rightarrow M$ accepts w $(w \in L_2, w \notin L_1)$ ✓

3) M_1 loops forever on w, M_2 accepts w
$\Rightarrow M$ loops forever on w $(w \in L_2, w \notin L_1)$ ✗

Very helpful for proving closure properties

Ex. Closure of recognizable languages under union. Suppose M_1 is a single-tape TM recognizing L_1, M_2 is a single-tape TM recognizing L_2

Design 2-tape TM recognizing $L_1 \cup L_2$. First attempt

M: On input w:

1) Copy w to tape 2
2) Run M_1 on tape 2. If accepts, accept. If rejects, go on.
3) Copy w back to tape 2
4) Run M_2 on tape 2. If accepts, accept. If rejects, reject.

What's wrong with this construction?

Why are Multi-Tape TMs Helpful?

To show a language is Turing-recognizable or decidable, it's enough to construct a multi-tape TM

Very helpful for proving closure properties

Ex. Closure of recognizable languages under union. Suppose M_1 is a single-tape TM recognizing L_1, M_2 is a single-tape TM recognizing L_2

Correct attempt: 3-tape TM

On input w:
1) Copy w to tape 2 and to tape 3
2) Repeat forever:
   Run M_1 on tape 2 for 1 step
   Run M_2 on tape 3 for 1 step
   If either accepts, accept.

Closure Properties

The Turing-decidable languages are closed under:

$L \quad \exists$ a TM $M \quad w \in L \iff M$ accepts w
$w \notin L \iff M$ rejects w (M has to halt on all inputs)

- Union
- Concatenation
- Star
- Intersection
- Reverse
- Complement ← HW 4

The Turing-recognizable languages are closed under:

$L \quad \exists$ a TM $M \quad w \in L \iff M$ accepts w
$w \notin L \iff M$ rejects w or M loops forever on w

- Union
- Concatenation
- Star
- Intersection
- Reverse

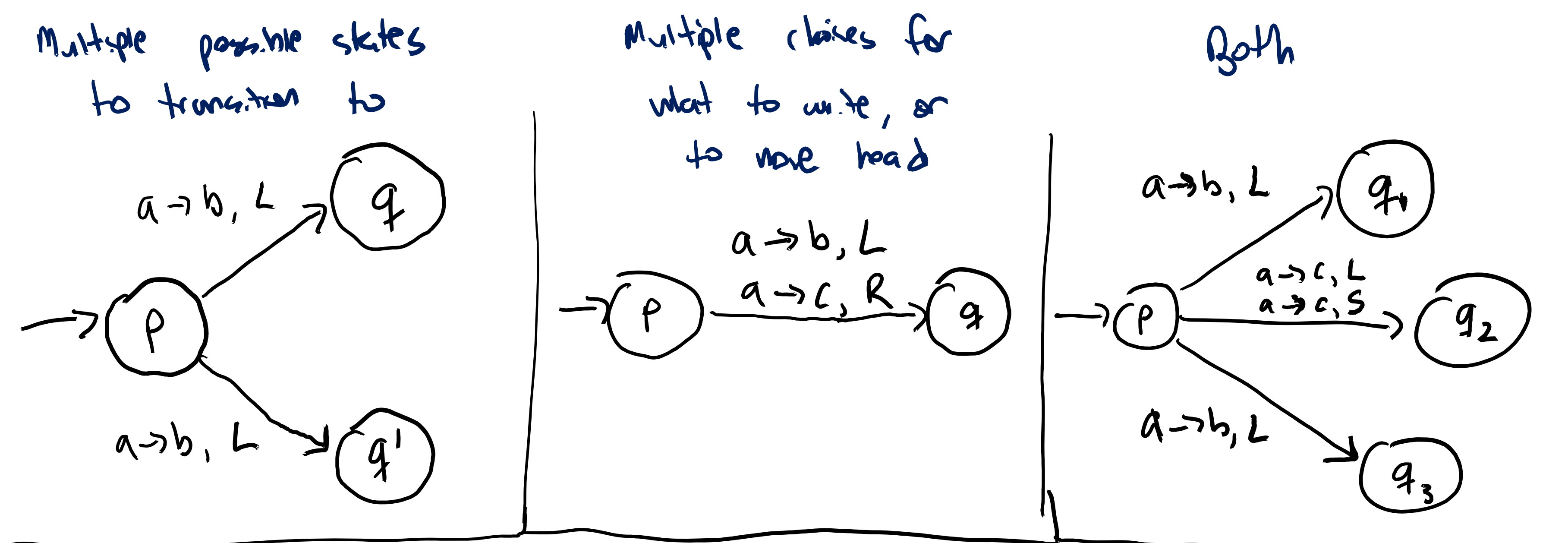
Nondeterministic TMs

Deterministic TM:

$\delta: Q \times \Gamma \to Q \times \Gamma \times \{L, R, S\}$

At any point in computation, may nondeterministically branch. Accepts iff there exists an accepting branch.

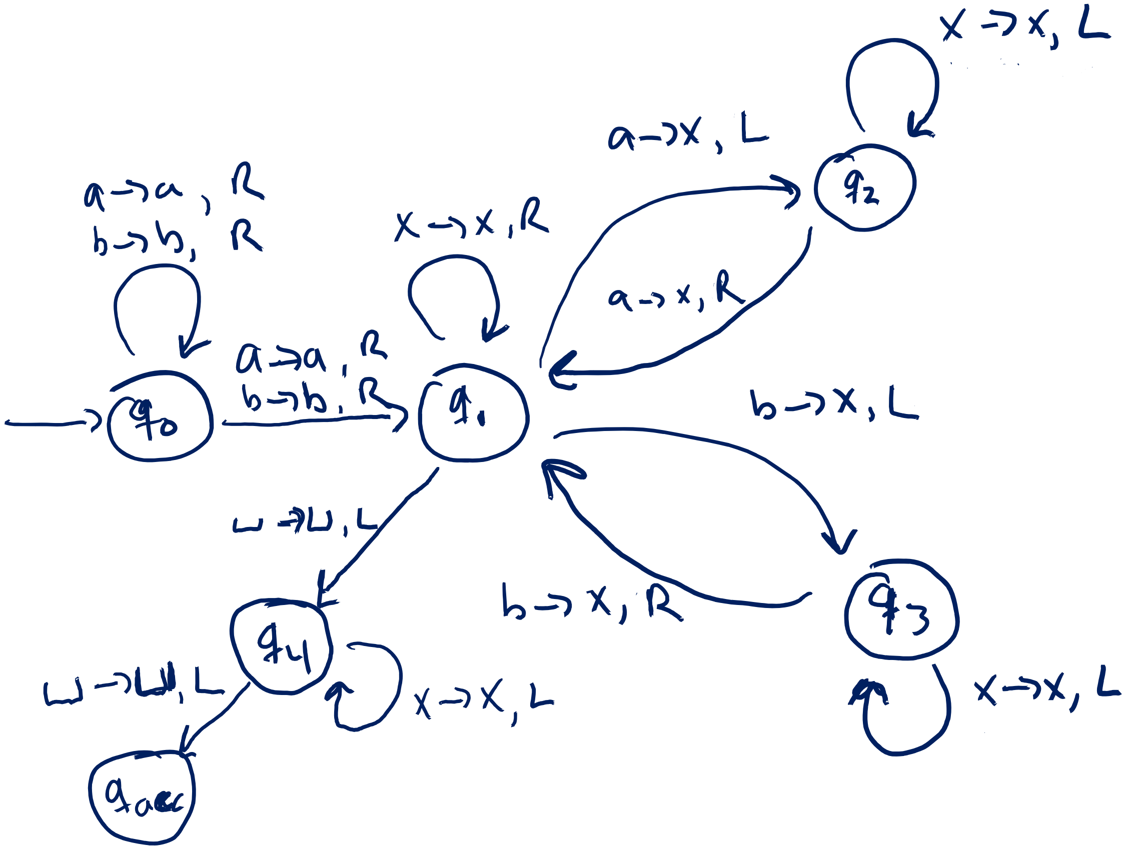
Transition function $\delta : Q \times \Gamma \to P(Q \times \Gamma \times \{L, R, S\})$



Nondeterministic TMs

Doubly infinite tape

At any point in computation, may nondeterministically branch. Accepts iff there exists an accepting branch.



Input is accepted since there is an accepting computation branch.

Input abba

q_0 abba
a q_0 bba
ab q_0 ba
abb q_0 a
abba q_0
reject

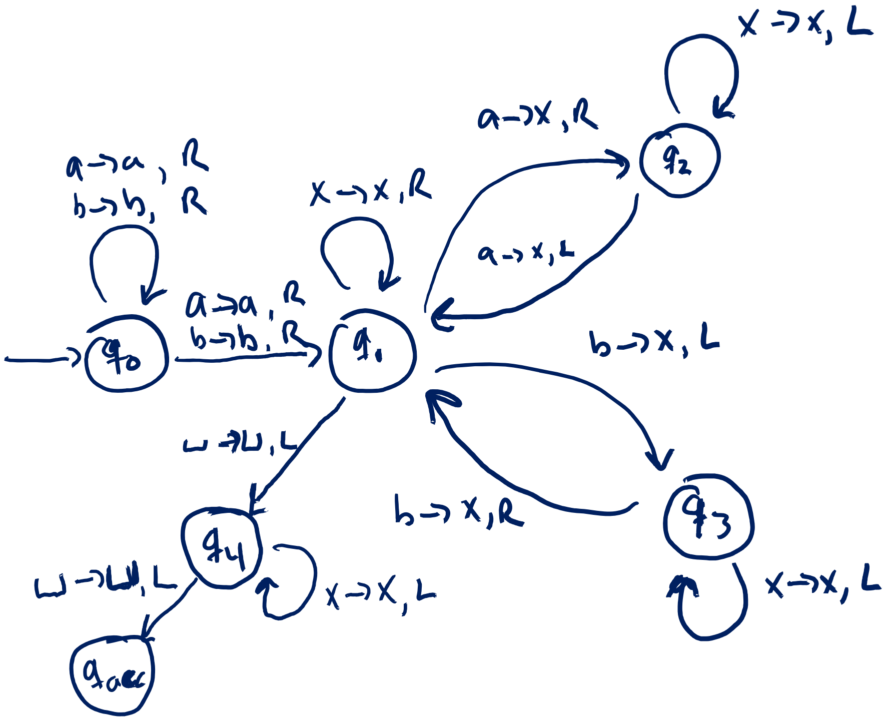
q_0 abba
a q_0 bba
ab q_1 ba
a q_3 bxa
ax q_1 xa
axx q_1 a
ax q_2 xx
a q_2 xxx
q_2 axxx
x q_1 xxx
...
xxxx q_1
xxx q_4 x
...
q_4 ⊔ xxxx
q_{accept} ⊔⊔ xxxx

Nondeterministic TMs

$w_1 w_2 \ldots w_n w_n w_{n-1} \ldots w_1$

~~a~~ ~~b~~ ~~b~~ ~~a~~

At any point in computation, may nondeterministically branch. Accepts iff there exists an accepting branch.



On input $w \in \{a, b\}^*$

1) Scan w left to right, at some point, nondeterministically go to step 2

2) Read next character, call this s
   Cross it off
   Move head left. If char matches s, cross it off, move head right until reaches a non-X
   Repeat

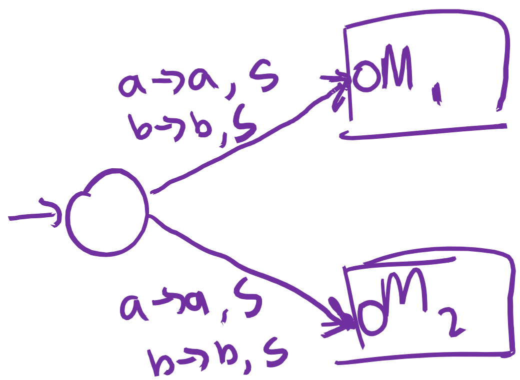
3) Once hit ⊔, check that string is all X's and accept if so. Else reject

What is the language of this NTM?

Nondeterministic TMs

Ex. Given TMs M_1 and M_2, construct an NTM recognizing $L(M_1) \cup L(M_2)$

alphabet $\{a, b\}$



+ merge accept / reject states

$\boxed{\text{Low-level}}$

On input w:

1) Nondeterministically either:
   a) Run M_1 on w, accept if accepts, reject if rejects
   b) Run M_2 on w, accept if accepts, reject if rejects

Nondeterministic TMs

$L =$

Ex. NTM for $\{w \mid w$ is a binary number representing the product of two positive integers $a, b\}$

≥ 2

On input w:

1) Nondeterministically guess $a \in \{2, \dots, w\}$, $b \in \{2, \dots, w\}$

2) Check $a \times b = w$: Accept if so, reject otherwise.

Analysis: If $w \in L$, $\exists\ \hat{a}, \hat{b} \in \{2, \dots, w\}$ s.t. $\hat{a} \times \hat{b} = w$

$\Rightarrow$ branch of computation where guessed $a = \hat{a}$, $b = \hat{b}$ leads to accept

If $w \notin L$, all choices of a, b will lead to $a \times b \neq w$

$\Rightarrow$ all branches lead to reject

Nondeterministic TMs

An NTM N accepts input w if when run on w it accepts on at least one computational branch

$$L(N) = \{w \mid N \text{ accepts input } w\}$$

An NTM N is a decider if on **every** input, it halts on **every** computational branch