BU CS 332 – Theory of Computation

Lecture 11:

- TM Variants and Closure Properties
- Church-Turing Thesis

Reading:

Sipser Ch 3.2

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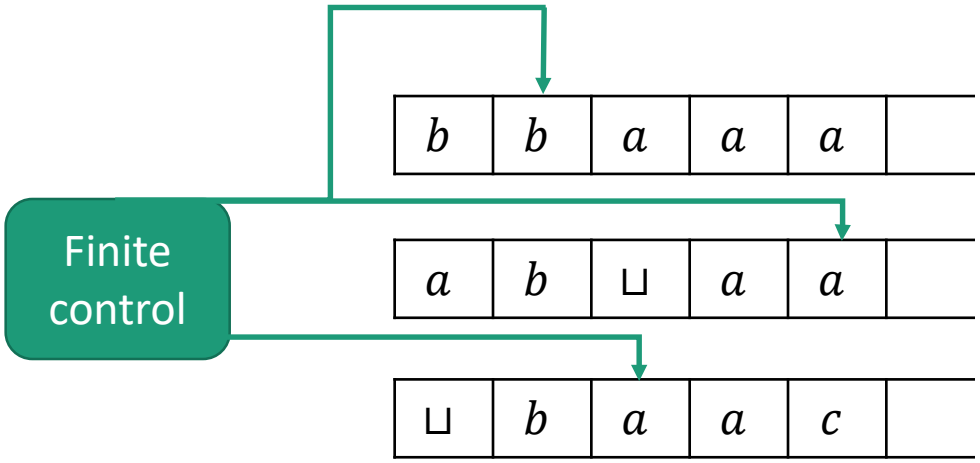
TM Variants

TMs are equivalent to…

- TMs with “stay put”
- TMs with 2-way infinite tapes
- Multi-tape TMs
- Nondeterministic TMs
- Random access TMs
- Enumerators
- Finite automata with access to an unbounded queue
- Primitive recursive functions
- Cellular automata

…

Multi-Tape TMs



Fixed number of tapes k (can't change during computation)

Transition function $\delta : Q \times \Gamma^k \rightarrow Q \times \Gamma^k \times \{L, R, S\}^k$

How to Simulate It

To show that a TM variant is no more powerful than the basic, single-tape TM:

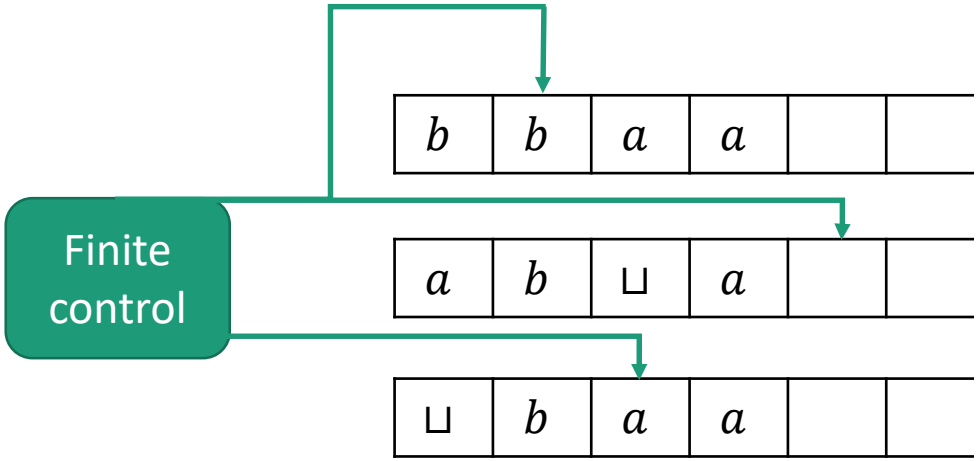
Show that if M is any variant machine, there exists a basic, single-tape TM M' that can simulate M

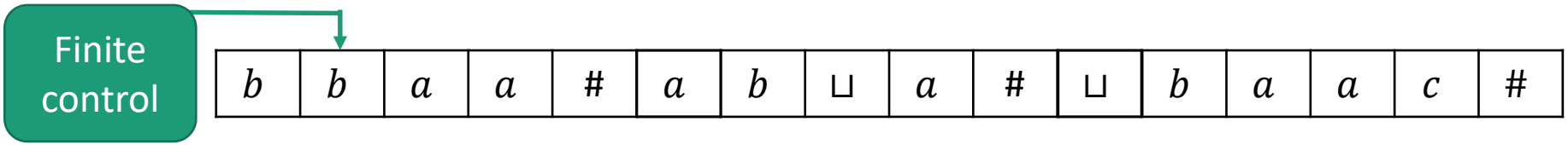
(Usual) parts of the simulation:

- Describe how to initialize the tapes of M' based on the input to M
- Describe how to simulate one step of M's computation using (possibly many steps of) M'

Multi-Tape TMs are Equivalent to Single-Tape TMs

Theorem: Every k-tape TM M with can be simulated by an equivalent single-tape TM M'





Simulating Multiple Tapes

Implementation-Level Description of M'

On input $w = w_1 w_2 \ldots w_n$

1. Format tape into $\# \dot{w}_1 w_2 \ldots w_n \# \dot{\sqcup} \# \dot{\sqcup} \# \ldots \#$
2. For each move of M:

   Scan left-to-right, finding current symbols

   Scan left-to-right, writing new symbols,

   Scan left-to-right, moving each tape head

   If a tape head goes off the right end, insert blank

   If a tape head goes off left end, move back right

Why are Multi-Tape TMs Helpful?

To show a language is Turing-recognizable or decidable, it's enough to construct a multi-tape TM

Often easier to construct multi-tape TMs

Ex. Decider for $\{a^i b^j | i > j\}$

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Very helpful for proving closure properties

Ex. Closure of recognizable languages under union. Suppose M_1 is a single-tape TM recognizing L_1, M_2 is a single-tape TM recognizing L_2

What's wrong with this construction?

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Closure Properties

The Turing-decidable languages are closed under:

- Union
- Concatenation
- Star
- Intersection
- Reverse
- Complement

The Turing-recognizable languages are closed under:

- Union
- Concatenation
- Star
- Intersection
- Reverse

Nondeterministic TMs

At any point in computation, may nondeterministically branch. Accepts iff there exists an accepting branch.

Transition function $\delta : Q \times \Gamma \rightarrow P(Q \times \Gamma \times \{L, R, S\})$

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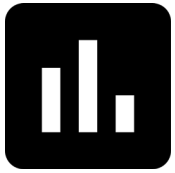


What is the language of this NTM?

Nondeterministic TMs

Ex. Given TMs M_1 and M_2, construct an NTM recognizing $L(M_1) \cup L(M_2)$

Nondeterministic TMs

Ex. NTM for $\{w \mid w$ is a binary number representing the product of two positive integers $a, b\}$

Nondeterministic TMs

An NTM N accepts input w if when run on w it accepts on at least one computational branch

$$L(N) = \{w \mid N \text{ accepts input } w\}$$

An NTM N is a decider if on **every** input, it halts on **every** computational branch

Nondeterministic TMs

Theorem: Every nondeterministic TM can be simulated by an equivalent deterministic TM

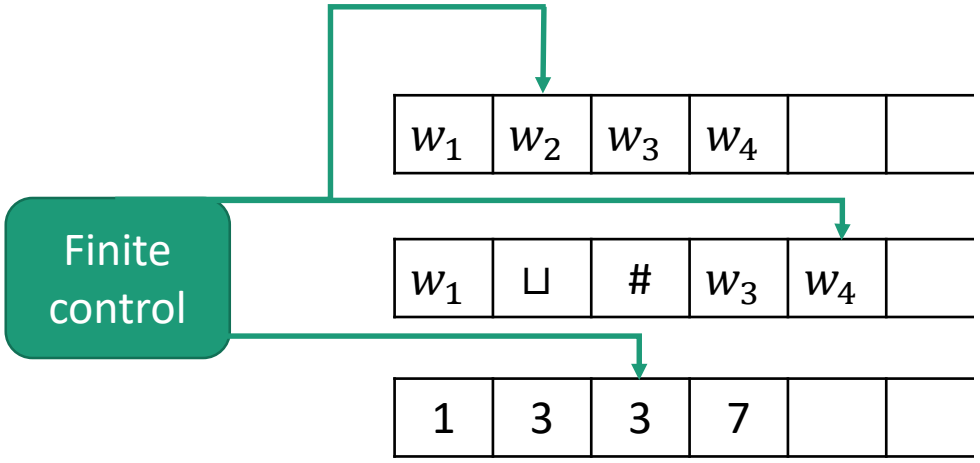
Proof idea:

Systematically try all 1-step computations, all 2-step computations, … and see if one of them accepts

Nondeterministic TMs

Theorem: Every nondeterministic TM has an equivalent deterministic TM

Proof idea: Simulate an NTM N using a 3-tape TM

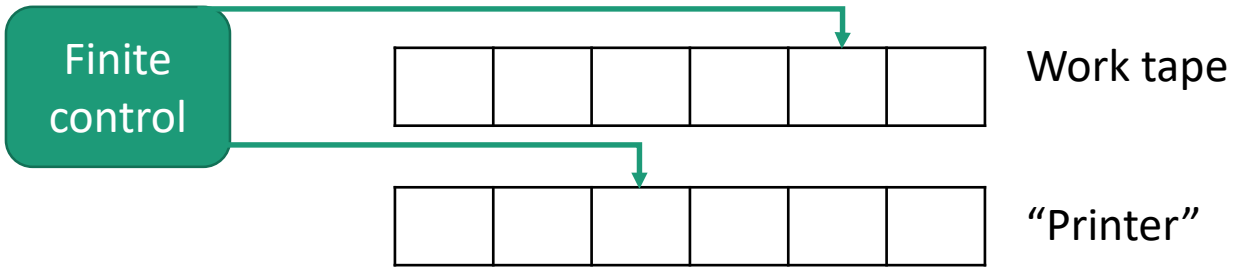


Input w to N (read-only)

Simulation tape (run N on w using nondeterministic choices from tape 3)

Address in computation tree

Enumerators



- Starts with two blank tapes
- Prints strings to printer

$L(E) = \{\text{strings eventually printed by } E\}$

- May never terminate (even if language is finite)
- May print the same string many times

Enumerator Example

What is the language enumerated by the following program?

"Write 1 to the work tape. Repeat the following forever: Multiply the binary number on the work tape by 2, and copy the result to the printer."

a) $\{x \mid x \text{ is a binary number } n^2 \text{ for some } n \geq 1\}$
b) $\{x \mid x \text{ is a binary number } 2^n \text{ for some } n \geq 0\}$
c) $\{x \mid x \text{ is a binary number } 2^{2^n} \text{ for some } n \geq 0\}$
d) None of the above

Enumerable = Turing-Recognizable

Theorem: A language is Turing-recognizable ⇔ some enumerator enumerates it

⇐ Start with an enumerator E for A and give a TM

Enumerable = Turing-Recognizable

Theorem: A language is Turing-recognizable ⇔ some enumerator enumerates it

⇒ Start with a TM M for A and give an enumerator

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Church-Turing Thesis

The equivalence of these models is a mathematical theorem

Church-Turing Thesis v1: The basic TM (hence all of these models) captures our intuitive notion of algorithms

Church-Turing Thesis v2: Any physically realizable model of computation can be simulated by the basic TM

The Church-Turing Thesis is not a mathematical statement!