BU CS 332 – Theory of Computation

Lecture 12:

- More on NTMs
- Church-Turing Thesis
- Decidable Languages

Reading:

Sipser Ch 3.2, 4.1

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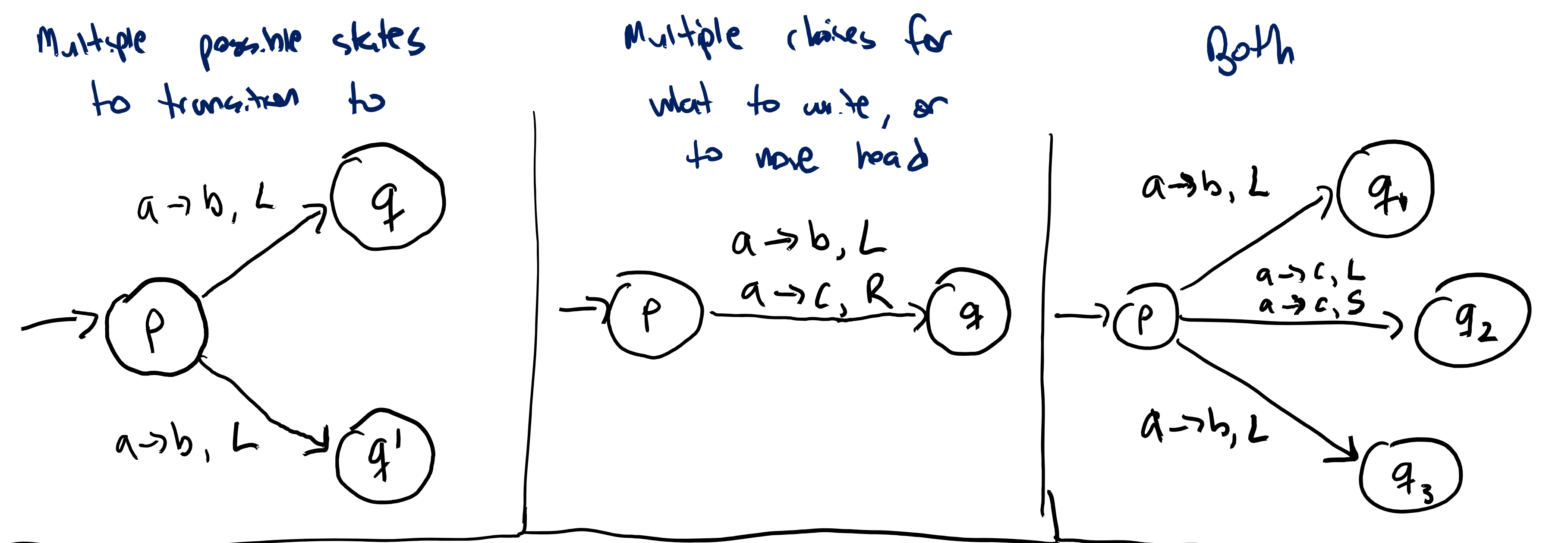
Nondeterministic TMs

Deterministic TM:

$\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R, S\}$

At any point in computation, may nondeterministically branch. Accepts iff there exists an accepting branch.

Transition function $\delta : Q \times \Gamma \to P(Q \times \Gamma \times \{L, R, S\})$



Nondeterministic TMs

An NTM N accepts input w if when run on w it accepts on at least one computational branch

$$L(N) = \{w \mid N \text{ accepts input } w\}$$

An NTM N is a decider if on **every** input, it halts on **every** computational branch

NTM recognizers can be simulated by deterministic TM recognizers

NTM deciders can be simulated by DTM deciders

NTM decider for L:

$w \in L \Rightarrow$ there exists a branch leading to accept on input w

$w \notin L \Rightarrow$ all branches lead to reject

Nondeterministic TMs

Ex. NTM decider for $L = \{w \mid w$ is a binary number representing the product of two integers $a, b \geq 2\}$

On input w:

1. Nondeterministically guess $a, b \in \{2, \dots, w\}$
2. Accept if $a \times b = w$, reject otherwise.

Proof of correctness:

If $w \in L$, there exist $\hat{a}, \hat{b}$ such that $\hat{a} \times \hat{b} = w$. Computation branch where we guessed $a = \hat{a}, b = \hat{b}$ accepts, so NTM accepts.

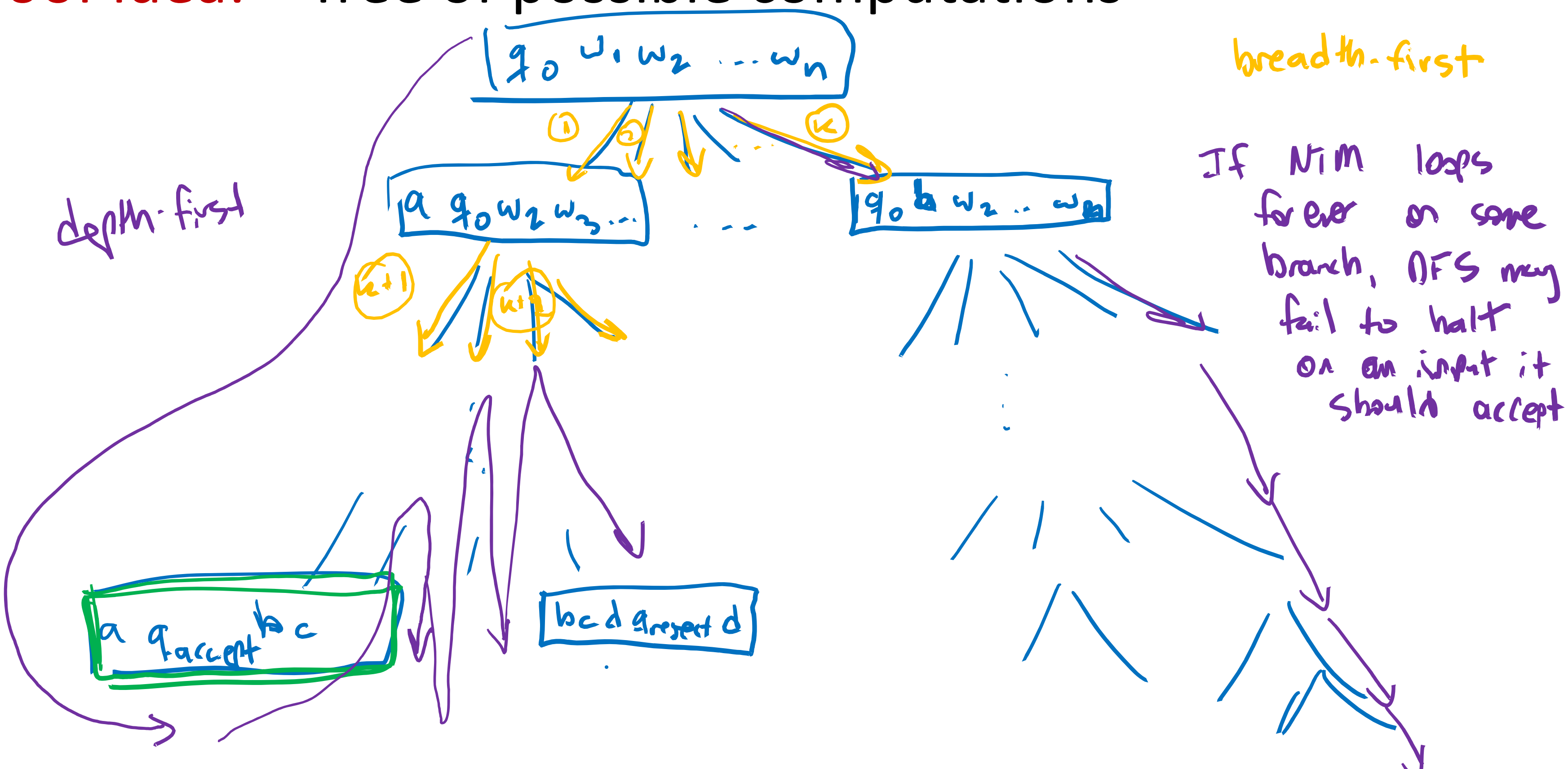
If $w \notin L$, all choices of a, b reject, so NTM does not accept.

This NTM is a decider because it halts on every computational branch

Simulating NTMs

Theorem: Every nondeterministic TM can be simulated by an equivalent deterministic TM

Proof idea: "Tree of possible computations"



Simulating NTMs

Which of the following algorithms is always appropriate for searching the tree of possible computations for an accepting configuration?

does work if NTM is a decider

a) Depth-first search: Explore as far as possible down each branch before backtracking

doesn't work if NTM could loop forever

b) Breadth-first search: Explore all the configurations at depth 1, then all the configurations at depth 2, etc.

always works

c) Either will always work

Simulating TMs

Theorem: Every nondeterministic TM can be simulated by an equivalent deterministic TM

Proof idea:

Breadth-first search:

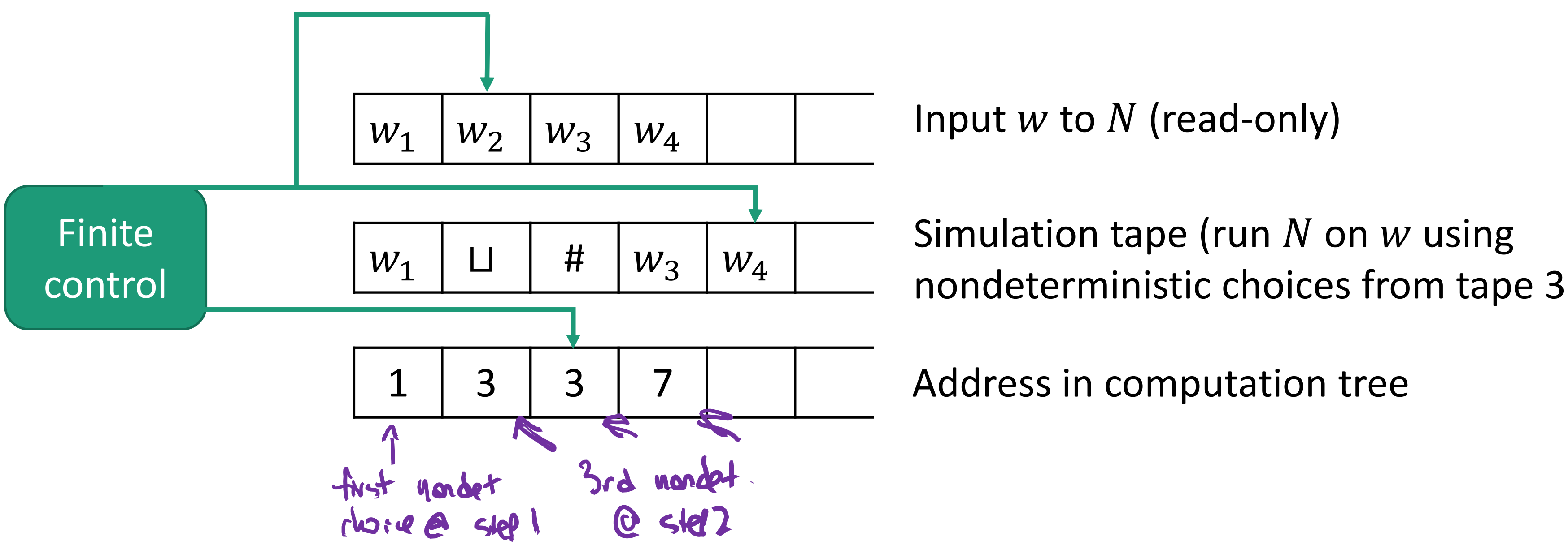
Systematically try all 1-step computations, all 2-step computations, … and see if one of them accepts

Nondeterministic TMs

Theorem: Every nondeterministic TM can be simulated by an equivalent deterministic TM

Proof idea: Simulate an NTM N using a 3-tape TM

(See Sipser for full description)



TMs are equivalent to…

- TMs with "stay put"
- TMs with 2-way infinite tapes
- Multi-tape TMs
- Nondeterministic TMs
- Random access TMs
- Enumerators
- Finite automata with access to an unbounded queue
- Primitive recursive functions
- Cellular automata

…

Church-Turing Thesis

The equivalence of these models is a mathematical theorem

Church-Turing Thesis v1: The basic TM (hence all of these models) captures our intuitive notion of algorithms

Normative, prescriptive, definitional

Church-Turing Thesis v2: Any physically realizable model of computation can be simulated by the basic TM

Empirical

The Church-Turing Thesis is not a mathematical statement! "Meta-mathematical"

Decidable Languages

1928 – The *Entscheidungsproblem*

The "Decision Problem"

"Turing machine" "mathematical statement"

Is there an algorithm which takes as input a formula (in first-order logic) and decides whether it's logically valid?

"true or false theorem"

Question: Can computers automate mathematicians

Question: How automatizable are the tasks we saw in language theory?

Questions about regular languages

- Given a DFA D and a string w, does D accept input w?
- Given a DFA D, does D recognize the empty language?
- Given DFAs D_1, D_2, do they recognize the same language?

(Same questions apply to NFAs, regexes)

Goal: Formulate each of these questions as a language, and decide them using Turing machines

Questions about regular languages

Design a TM which takes as input a DFA D and a string w, and determines whether D accepts w Test if $w \in L(D)$

How should the input to this TM be represented? Encoding to String!

Let $D = (Q, \Sigma, \delta, q_0, F)$. List each component of the tuple separated by #

- Represent Q by ,-separated binary strings
- Represent Σ by ,-separated binary strings
- Represent $\delta : Q \times \Sigma \rightarrow Q$ by a ,-separated list of triples (p, a, q), …

Denote the encoding of D, w by $\langle D, w \rangle$

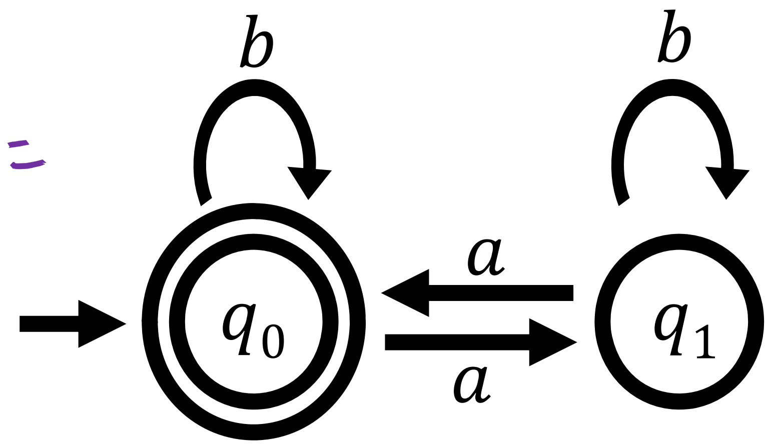
Example

$q_0 : 0 \quad , \quad q_1 : 1$

$\Sigma_1 = \{a, b\} \qquad a : 00$
$\qquad\qquad\qquad\qquad b : 11$

$\delta(q_0, a) = q_1 , \ldots$

$D =$



$$\langle D \rangle = \overbrace{0,1}^{Q} \# \overbrace{00,11}^{\Sigma_1} \# \overbrace{(0,00,1),(0,11,0),(1,00,0),(1,11,1)}^{\delta} \# \underbrace{0}_{q_0} \# \underbrace{0}_{F}$$

Representation independence

Computability (i.e., decidability and recognizability) is **not** affected by the precise choice of encoding

Let [·] be a different encoding scheme

Why? A TM can always convert between different (reasonable) encodings

Decide if [D, w] represents DFA D which accepts w as follows:

1) Convert encoding [D, w] to ⟨D, w⟩
2) Given ⟨D, w⟩, determine if D accepts w

We'll take ⟨ ⟩ to mean "any reasonable encoding"

A “universal” algorithm for recognizing regular languages

$A_{\text{DFA}} = \{\langle D, w\rangle \mid \text{DFA } D \text{ accepts } w\}$

Theorem: A_{DFA} is decidable

Question: Does D accept w?

$\Longleftrightarrow$

Language: $\langle D, w\rangle \overset{?}{\in} A_{DFA}$.

Proof: Define a 3-tape TM M on input $\langle D, w\rangle$:

1. Check if $\langle D, w\rangle$ is a valid encoding (reject if not)

   Feel free to omit in your description

2. Simulate D on w, i.e.,
   - Tape 2: Maintain w and head location of D

   - Tape 3: Maintain state of D, update according to δ

3. Accept if D ends in an accept state, reject otherwise

Other decidable languages

$A_{\text{DFA}} = \{\langle D, w\rangle \mid \text{DFA } D \text{ accepts } w\}$

$A_{\text{NFA}} = \{\langle N, w\rangle \mid \text{NFA } N \text{ accepts } w\}$

$A_{\text{REX}} = \{\langle R, w\rangle \mid \text{regular expression } R \text{ generates } w\}$

NFA Acceptance

Does NFA N accept w?

Which of the following describes a **decider** for $A_{\text{NFA}} = \{\langle N, w\rangle \mid \text{NFA } N \text{ accepts } w\}$?

a) Using a deterministic TM, simulate N on w, always making the first nondeterministic choice at each step. Accept if it accepts, and reject otherwise.

Doesn't explore all possible branches of N on w

b) Using a deterministic TM, simulate all possible choices of N on w for 1 step of computation, 2 steps of computation, etc. Accept whenever some simulation accepts.

Recognizer, but might not halt

$N =$  $w = 0$

Naively, if $\langle N, w\rangle \notin A_{\text{NFA}}$, might loop forever

c) Convert N to an equivalent DFA M. Simulate M on w, accept if it accepts, and reject otherwise. ✓

via subset construction

Regular Languages are Decidable

Theorem: Every regular language L is decidable

Proof 1: If L is regular, it is recognized by a DFA D. Convert this DFA to a TM M. Then M decides L.

L regular $\Rightarrow$ L decidable

Proof 2: If L is regular, it is recognized by a DFA D. The following TM M_D decides L.

On input w:

1. Run the decider for A_{DFA} on input $\langle D, w \rangle$
2. Accept if the decider accepts; reject otherwise

Analysis:

- If $w \in L$, D accepts w $\Rightarrow$ $\langle D, w \rangle \in A_{DFA}$ $\Rightarrow$ decider accepts
- If $w \notin L$, D rejects w $\Rightarrow$ $\langle D, w \rangle \notin A_{DFA}$ $\Rightarrow$ decider rejects

Classes of Languages