BU CS 332 – Theory of Computation

Lecture 12:

- More on NTMs
- Church-Turing Thesis
- Decidable Languages

Reading:

Sipser Ch 3.2, 4.1

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Nondeterministic TMs

At any point in computation, may nondeterministically branch. Accepts iff there exists an accepting branch.

Transition function $\delta : Q \times \Gamma \to P(Q \times \Gamma \times \{L, R, S\})$

Nondeterministic TMs

An NTM N accepts input w if when run on w it accepts on at least one computational branch

$$L(N) = \{w \mid N \text{ accepts input } w\}$$

An NTM N is a decider if on **every** input, it halts on **every** computational branch

Nondeterministic TMs

Ex. NTM decider for $L = \{w \mid w \text{ is a binary number representing the product of two integers } a, b \geq 2\}$

On input w:

1. Nondeterministically guess $a, b \in \{2, \ldots, w\}$
2. Accept if $a \times b = w$, reject otherwise.

Proof of correctness:

If $w \in L$, there exist $\hat{a}, \hat{b}$ such that $\hat{a} \times \hat{b} = w$. Computation branch where we guessed $a = \hat{a}, b = \hat{b}$ accepts, so NTM accepts.

If $w \notin L$, all choices of a, b reject, so NTM does not accept.

Simulating NTMs

Theorem: Every nondeterministic TM can be simulated by an equivalent deterministic TM

Proof idea: “Tree of possible computations”

Simulating NTMs

Which of the following algorithms is always appropriate for searching the tree of possible computations for an accepting configuration?

a) Depth-first search: Explore as far as possible down each branch before backtracking

b) Breadth-first search: Explore all the configurations at depth 1, then all the configurations at depth 2, etc.

c) Either will always work

Simulating TMs

Theorem: Every nondeterministic TM can be simulated by an equivalent deterministic TM

Proof idea:

Breadth-first search:

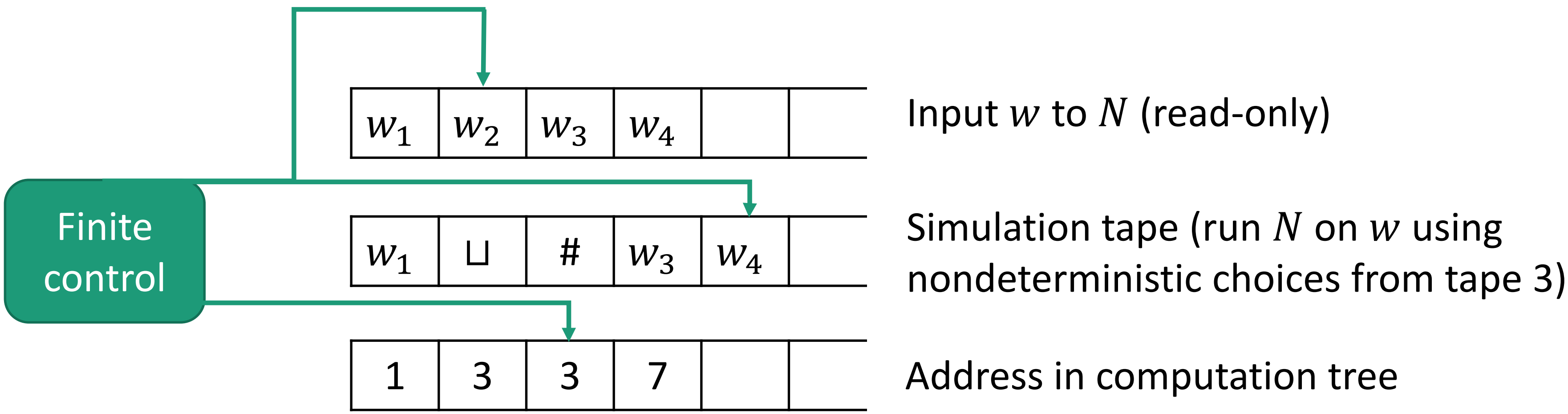
Systematically try all 1-step computations, all 2-step computations, … and see if one of them accepts

Nondeterministic TMs

Theorem: Every nondeterministic TM can be simulated by an equivalent deterministic TM

Proof idea: Simulate an NTM N using a 3-tape TM

(See Sipser for full description)



TMs are equivalent to…

- TMs with “stay put”
- TMs with 2-way infinite tapes
- Multi-tape TMs
- Nondeterministic TMs
- Random access TMs
- Enumerators
- Finite automata with access to an unbounded queue
- Primitive recursive functions
- Cellular automata

…

Church-Turing Thesis

The equivalence of these models is a mathematical theorem

Church-Turing Thesis v1: The basic TM (hence all of these models) captures our intuitive notion of algorithms

Church-Turing Thesis v2: Any physically realizable model of computation can be simulated by the basic TM

The Church-Turing Thesis is not a mathematical statement!

Decidable Languages

1928 – The *Entscheidungsproblem*

The "Decision Problem"

Is there an algorithm which takes as input a formula (in first-order logic) and decides whether it's logically valid?

Questions about regular languages

- Given a DFA D and a string w, does D accept input w?
- Given a DFA D, does D recognize the empty language?
- Given DFAs D_1, D_2, do they recognize the same language?

(Same questions apply to NFAs, regexes)

Goal: Formulate each of these questions as a language, and decide them using Turing machines

Questions about regular languages

Design a TM which takes as input a DFA D and a string w, and determines whether D accepts w

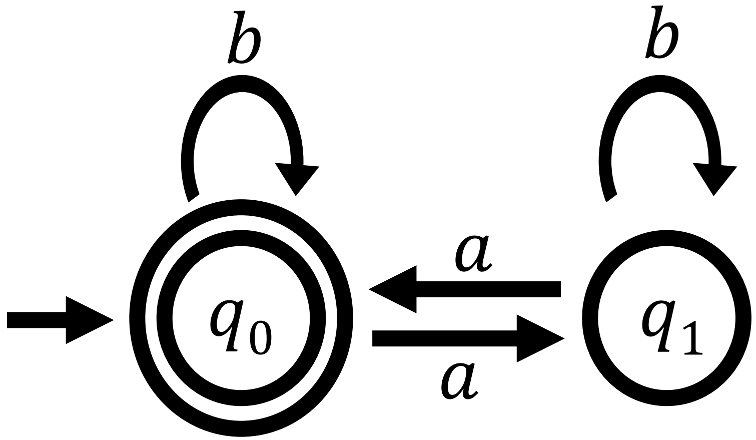
How should the input to this TM be represented?

Let $D = (Q, \Sigma, \delta, q_0, F)$. List each component of the tuple separated by #

- Represent Q by ,-separated binary strings
- Represent Σ by ,-separated binary strings
- Represent $\delta : Q \times \Sigma \rightarrow Q$ by a ,-separated list of triples (p, a, q), …

Denote the encoding of D, w by $\langle D, w \rangle$

Example



Representation independence

Computability (i.e., decidability and recognizability) is **not** affected by the precise choice of encoding

Why? A TM can always convert between different (reasonable) encodings

We'll take $\langle \quad \rangle$ to mean "any reasonable encoding"

A “universal” algorithm for recognizing regular languages

$A_{\text{DFA}} = \{\langle D, w \rangle \mid \text{DFA } D \text{ accepts } w\}$

Theorem: A_{DFA} is decidable

Proof: Define a 3-tape TM M on input $\langle D, w \rangle$:

1. Check if $\langle D, w \rangle$ is a valid encoding (reject if not)
2. Simulate D on w, i.e.,
   - Tape 2: Maintain w and head location of D
   - Tape 3: Maintain state of D, update according to δ
3. Accept if D ends in an accept state, reject otherwise

Other decidable languages

$A_{\mathrm{DFA}} = \{\langle D, w\rangle \mid \text{DFA } D \text{ accepts } w\}$

$A_{\mathrm{NFA}} = \{\langle N, w\rangle \mid \text{NFA } N \text{ accepts } w\}$

$A_{\mathrm{REX}} = \{\langle R, w\rangle \mid \text{regular expression } R \text{ generates } w\}$

NFA Acceptance

Which of the following describes a **decider** for $A_{\mathrm{NFA}} = \{\langle N, w\rangle \mid \text{NFA } N \text{ accepts } w\}$?

a) Using a deterministic TM, simulate N on w, always making the first nondeterministic choice at each step. Accept if it accepts, and reject otherwise.

b) Using a deterministic TM, simulate all possible choices of N on w for 1 step of computation, 2 steps of computation, etc. Accept whenever some simulation accepts.

c) Convert N to an equivalent DFA M. Simulate M on w, accept if it accepts, and reject otherwise.

Regular Languages are Decidable

Theorem: Every regular language L is decidable

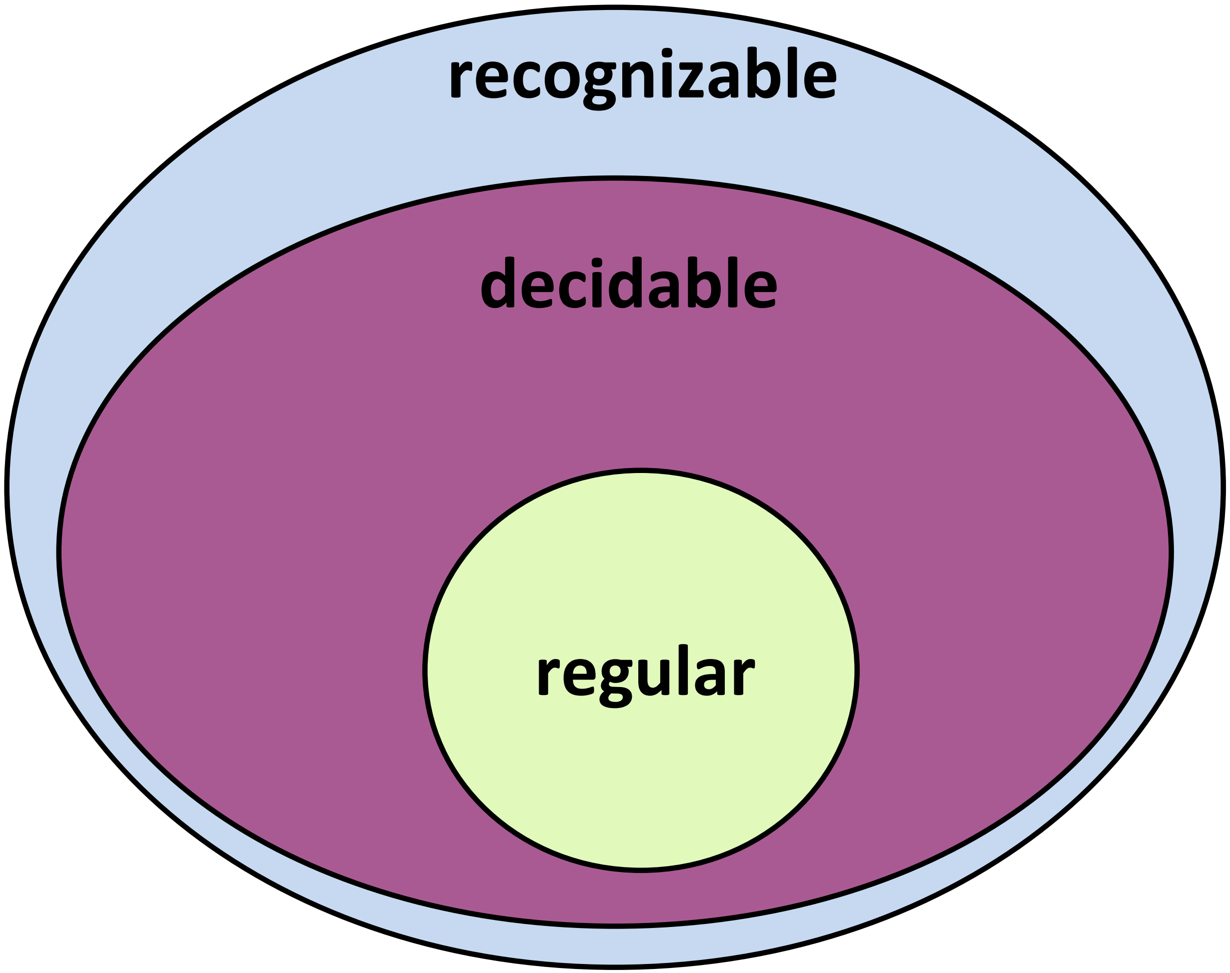
Proof 1: If L is regular, it is recognized by a DFA D. Convert this DFA to a TM M. Then M decides L.

Proof 2: If L is regular, it is recognized by a DFA D. The following TM M_D decides L.

On input w:

1. Run the decider for A_{DFA} on input $\langle D, w\rangle$
2. Accept if the decider accepts; reject otherwise

Classes of Languages



Emptiness Testing

$E_{\mathrm{DFA}} = \{\langle D \rangle \mid D \text{ is a DFA that recognizes the empty language}\}$

Decidability of E_{DFA}

Theorem: $E_{\text{DFA}} = \{\langle D \rangle \mid D \text{ is a DFA that recognizes } \emptyset\}$ is decidable

Proof: The following TM decides E_{DFA}

On input $\langle D \rangle$, where D is a DFA with k states:

1. Perform k steps of breadth-first search on state diagram of D to determine if an accept state is reachable from the start state
2. Accept if an accept state reachable; reject otherwise

E_{DFA} Example

New Deciders from Old

$EQ_{\mathrm{DFA}} = \{\langle D_1, D_2\rangle \mid D_1, D_2 \text{ are DFAs and } L(D_1) = L(D_2)\}$

Theorem: EQ_{DFA} is decidable

Proof: The following TM decides EQ_{DFA}

On input $\langle D_1, D_2\rangle$, where $\langle D_1, D_2\rangle$ are DFAs:

1. Construct a DFA D that recognizes the **symmetric difference** $L(D_1) \triangle L(D_2)$
2. Run the decider for E_{DFA} on $\langle D\rangle$ and return its output

Symmetric Difference

$$A \triangle B = \{w \mid w \in A \text{ or } w \in B \text{ but not both}\}$$