Lecture 13:

• Decidable Languages
• Universal TM
• Countability and Diagonalization

Reading:
Sipser Ch 4.1, 4.2

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A “universal” algorithm for recognizing regular languages

\[ A_{\text{DFA}} = \{ \langle D, w \rangle \mid \text{DFA } D \text{ accepts } w \} \]

Theorem: \( A_{\text{DFA}} \) is decidable

Proof: Define a 3-tape TM \( M \) on input \( \langle D, w \rangle \):

1. Check if \( \langle D, w \rangle \) is a valid encoding (reject if not)
2. Simulate \( D \) on \( w \), i.e.,
   - Tape 2: Maintain \( w \) and head location of \( D \)
   - Tape 3: Maintain state of \( D \), update according to \( \delta \)
3. Accept if \( D \) ends in an accept state, reject otherwise
More Decidable Languages: Emptiness Testing

**Theorem:** $E_{\text{DFA}} = \{ (D) \mid D$ is a DFA that recognizes $\emptyset$ $\}$ is decidable

**Proof:** The following TM decides $E_{\text{DFA}}$

On input $(D)$, where $D$ is a DFA with $k$ states:

1. Perform $k$ steps of breadth-first search on state diagram of $D$ to determine if an accept state is reachable from the start state
2. **Reject** if a DFA accept state is reachable; **accept** otherwise
$E_{DFA}$ Example

$L(N) \neq \emptyset$ (e.g. $11 \in L(N)$)

Breadth-first search:
- $q_0$
- $q_1$
- $q_2$
- $q_3$
- $q_4$
- $q_5$

Notice $q_5$ is an accept state

Reject. Conclude $L(N) \neq \emptyset$
New Deciders from Old: Equality Testing

\[ EQ_{\text{DFA}} = \{ \langle D_1, D_2 \rangle \mid D_1, D_2 \text{ are DFAs and } L(D_1) = L(D_2) \} \]

**Theorem:** \( EQ_{\text{DFA}} \) is decidable

**Proof:** The following TM decides \( EQ_{\text{DFA}} \)

On input \( \langle D_1, D_2 \rangle \), where \( \langle D_1, D_2 \rangle \) are DFAs:

1. Construct a DFA \( D \) that recognizes the **symmetric difference** \( L(D_1) \triangle L(D_2) = \{ w \mid w \text{ is in exactly one of } L(D_1) \text{ or } L(D_2) \} \)
2. Run the decider for \( E_{\text{DFA}} \) on \( \langle D \rangle \) and return its output

**Analysis (correctness):**
- If \( \langle D_1, D_2 \rangle \in EQ_{\text{DFA}} \), then \( L(D_1) = L(D_2) \) \( \Rightarrow \) \( L(D_1) \triangle L(D_2) = \emptyset \) \( \Rightarrow \) \( L(\emptyset) = \emptyset \Rightarrow \) accepts
- If \( \langle D_1, D_2 \rangle \notin EQ_{\text{DFA}} \), then \( L(D_1) \neq L(D_2) \) \( \Rightarrow \) \( L(D_1) \triangle L(D_2) \neq \emptyset \) \( \Rightarrow \) \( L(\emptyset) \neq \emptyset \Rightarrow \) rejects
Symmetric Difference

\[ A \triangle B = \{ w \mid w \in A \text{ or } w \in B \text{ but not both} \} \]

\[
A \triangle B = \left( A \setminus B \right) \cup \left( B \setminus A \right) \\
= \left( A \cap \overline{B} \right) \cup \left( B \cap \overline{A} \right) \\
= \left( \overline{A \cup B} \right) \cup \left( \overline{B \cup A} \right)
\]

If \( A = L(0,1) \) and \( B = L(0^2) \) then construct DFA for \( L(0) \triangle L(0^2) \) using complement, union, and subset constructions.
Meta-Computational Languages

\[ A_{\text{DFA}} = \{ \langle D, w \rangle \mid \text{DFA } D \text{ accepts } w \} \]
\[ A_{\text{TM}} = \{ \langle M, w \rangle \mid \text{TM } M \text{ accepts } w \} \]

\[ E_{\text{DFA}} = \{ \langle D \rangle \mid \text{DFA } D \text{ recognizes the empty language } \emptyset \} \]
\[ E_{\text{TM}} = \{ \langle M \rangle \mid \text{TM } M \text{ recognizes the empty language } \emptyset \} \]

\[ E_{Q_{\text{DFA}}} = \{ \langle D_1, D_2 \rangle \mid D_1 \text{ and } D_2 \text{ are DFAs, } L(D_1) = L(D_2) \} \]
\[ E_{Q_{\text{TM}}} = \{ \langle M_1, M_2 \rangle \mid M_1 \text{ and } M_2 \text{ are TMs, } L(M_1) = L(M_2) \} \]
The Universal Turing Machine

$A_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM that accepts input } w \}$

Theorem: $A_{TM}$ is Turing-recognizable

The following “universal TM” $U$ recognizes $A_{TM}$

On input $\langle M, w \rangle$:

1. Simulate running $M$ on input $w$
2. If $M$ accepts, accept. If $M$ rejects, reject.

1) If $M$ accepts $w$, then $U(\langle M, w \rangle)$ accepts $\checkmark$
2) If $M$ rejects $w$, then $U(\langle M, w \rangle)$ rejects $\checkmark$
3) If $M$ loops on $w$, then $U(\langle M, w \rangle)$ loops

($w \in \mathcal{L}(M)$, $\langle M, w \rangle \notin A_{TM}$)
Universal TM and $A_{TM}$

Why is the Universal TM not a decider for $A_{TM}$?

The following "universal TM" $U$ recognizes $A_{TM}$

On input $\langle M, w \rangle$:
1. Simulate running $M$ on input $w$
2. If $M$ accepts, accept. If $M$ rejects, reject.

a) It may reject inputs $\langle M, w \rangle$ where $M$ accepts $w$

b) It may accept inputs $\langle M, w \rangle$ where $M$ rejects $w$

c) It may loop on inputs $\langle M, w \rangle$ where $M$ loops on $w$

d) It may loop on inputs $\langle M, w \rangle$ where $M$ rejects $w$
More on the Universal TM

"It is possible to invent a single machine which can be used to compute any computable sequence. If this machine $U$ is supplied with a tape on the beginning of which is written the S.D ["standard description"] of some computing machine $M$, then $U$ will compute the same sequence as $M$.”

- Turing, “On Computable Numbers...” 1936

• Foreshadowed general-purpose programmable computers

• No need for specialized hardware: Virtual machines as software

Harvard architecture:
Separate instruction and data pathways

von Neumann architecture:
Programs can be treated as data
Undecidability

$A_{TM}$ is Turing-recognizable via Universal TM $U$

...but it turns out $A_{TM}$ (and $E_{TM}, E_{Q_{TM}}$) is undecidable

i.e., computers cannot solve these problems no matter how much time they are given
Countability and Diagonalization
What’s your intuition?

Which of the following sets is the “biggest”?

a) The natural numbers: \( \mathbb{N} = \{1, 2, 3, \ldots\} \)

b) The even numbers: \( E = \{2, 4, 6, \ldots\} \)

\( \text{relabel: } 1, 2, 3, \ldots \)

c) The positive powers of 2: \( POW2 = \{2, 4, 8, 16, \ldots\} \)

\( \text{relabel: } 1, 2, 3, 4, \ldots \)

\( (\log_2) \)

d) They all have the same size \( \leq \)
Set Theory Review

A function $f: A \to B$ is

- **1-to-1 (injective)** if $f(a) \neq f(a')$ for all $a \neq a'$

- **onto (surjective)** if for all $b \in B$, there exists $a \in A$ such that $f(a) = b$

- a correspondence (bijective) if it is 1-to-1 and onto, i.e., every $b \in B$ has a unique $a \in A$ with $f(a) = b$
How can we compare sizes of infinite sets?

**Definition:** Two sets have the same size if there is a bijection between them.

\[ \exists \; x, y, z \in S \]
\[ \downarrow \quad \downarrow \quad \downarrow \]
\[ a, b, c \in \mathbb{N} \]

\[ f(x) = a \]
\[ f(y) = b \]
\[ f(z) = c \]

A set is **countable** if

- it is a finite set, or
- it has the same size as \( \mathbb{N} \), the set of natural numbers

(i.e. \( \exists \) a bijection \( f : \mathbb{N} \rightarrow S \)) "countably infinite"
Examples of countable sets

• $\emptyset$
• $\{0,1\}$
• $\{0, 1, 2, \ldots, 8675309\}$

\[
\begin{align*}
\text{finite} & \Rightarrow \text{countable} \\
\{ E = \{2, 4, 6, 8, \ldots \} \quad f(i) = 2^i \\
\text{SQUARES} = \{1, 4, 9, 16, 25, \ldots \} \quad f(i) = i^2 \\
\text{POW2} = \{2, 4, 8, 16, 32, \ldots \} \quad f(i) = 2^i \\
\end{align*}
\]

$|E| = |\text{SQUARES}| = |\text{POW2}| = |\mathbb{N}|$

$\Rightarrow$ countable
How to show that $\mathbb{N} \times \mathbb{N}$ is countable?

Construct bijection $f: \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$

$(1, 1) \rightarrow (2, 1) \rightarrow (3, 1) \rightarrow (4, 1) \rightarrow (5, 1) \rightarrow \ldots$

$(1, 2) \rightarrow (2, 2) \rightarrow (3, 2) \rightarrow (4, 2) \rightarrow (5, 2) \rightarrow \ldots$

$(1, 3) \rightarrow (2, 3) \rightarrow (3, 3) \rightarrow (4, 3) \rightarrow (5, 3) \rightarrow \ldots$

$(1, 4) \rightarrow (2, 4) \rightarrow (3, 4) \rightarrow (4, 4) \rightarrow (5, 4) \rightarrow \ldots$

$(1, 5) \rightarrow (2, 5) \rightarrow (3, 5) \rightarrow (4, 5) \rightarrow (5, 5) \rightarrow \ldots$
How to argue that a set $S$ is countable

• Describe how to “list” the elements of $S$, usually in stages:

**Ex:** Stage 1) List all pairs $(x, y)$ such that $x + y = 2$ \( \binom{1}{1} \)

Stage 2) List all pairs $(x, y)$ such that $x + y = 3$ \( \binom{2}{1}, \binom{1}{2} \)

... 

Stage $n$) List all pairs $(x, y)$ such that $x + y = n + 1$ 

... 

• Argue that every element of $S$ appears in the list

**Ex:** Any $(x, y) \in \mathbb{N} \times \mathbb{N}$ will be listed in stage $x + y - 1$

• Define the bijection $f: \mathbb{N} \rightarrow S$ by $f(n) =$ the $n$’th element in this list (ignoring duplicates if needed)
Subsets of countable sets

If $A$ and $B$ are sets with $A \subseteq B$ ($A$ is a subset of $B$), which of the following statements are true?

a) If $A$ is countable, then $B$ is countable
b) If $B$ is countable, then $A$ is countable

C) Both are true
d) Neither is true
More examples of countable sets

• \{0,1\}^*

• \{\langle M \rangle \mid M \text{ is a Turing machine}\} \subseteq \{0,1\}^*

• \(\mathbb{Q} = \{\text{rational numbers}\}\) use the construction for \(\mathbb{N} \times \mathbb{N}\)

So what *isn’t* countable?
Cantor’s Diagonalization Method

- Invented set theory
- Defined countability, uncountability, cardinal and ordinal numbers, ...

Some praise for his work:

“Scientific charlatan...renegade...corruptor of youth”
–L. Kronecker

“Set theory is wrong...utter nonsense...laughable”
–L. Wittgenstein
Uncountability of the reals

**Theorem:** The real interval $(0, 1)$ is uncountable.

**Proof:** Assume for the sake of contradiction it were countable, and let $f : \mathbb{N} \to (0,1)$ be a bijection

<table>
<thead>
<tr>
<th>$n$</th>
<th>$f(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$0.d_1^1 d_2^1 d_3^1 d_4^1 d_5^1 \ldots$</td>
</tr>
<tr>
<td>2</td>
<td>$0.d_1^2 d_2^2 d_3^2 d_4^2 d_5^2 \ldots$</td>
</tr>
<tr>
<td>3</td>
<td>$0.d_1^3 d_2^3 d_3^3 d_4^3 d_5^3 \ldots$</td>
</tr>
<tr>
<td>4</td>
<td>$0.d_1^4 d_2^4 d_3^4 d_4^4 d_5^4 \ldots$</td>
</tr>
<tr>
<td>5</td>
<td>$0.d_1^5 d_2^5 d_3^5 d_4^5 d_5^5 \ldots$</td>
</tr>
</tbody>
</table>

Construct $b \in (0,1)$ which does not appear in this table – contradiction!

$b = 0.b_1 b_2 b_3 \ldots$ where $b_n \neq d_n^n$ (digit $n$ of $f(n)$)