BU CS 332 – Theory of Computation

Lecture 13:

- Decidable Languages
- Universal TM
- Countability and Diagonalization

Reading:

Sipser Ch 4.1, 4.2

Mark Bun

March 8, 2021

A "universal" algorithm for recognizing regular languages

$A_{\mathrm{DFA}} = \{\langle D, w \rangle \mid \text{DFA } D \text{ accepts } w\}$

Theorem: A_{DFA} is decidable

Proof: Define a 3-tape TM M on input $\langle D, w \rangle$:

1. Check if $\langle D, w \rangle$ is a valid encoding (reject if not)
2. Simulate D on w, i.e.,
   - Tape 2: Maintain w and head location of D
   - Tape 3: Maintain state of D, update according to δ
3. Accept if D ends in an accept state, reject otherwise

More Decidable Languages: Emptiness Testing

Theorem: $E_{\text{DFA}} = \{\langle D \rangle \mid D \text{ is a DFA that recognizes } \emptyset\}$ is decidable

Proof: The following TM decides E_{DFA}

On input $\langle D \rangle$, where D is a DFA with k states:

1. Perform k steps of breadth-first search on state diagram of D to determine if an accept state is reachable from the start state
2. Reject if a DFA accept state is reachable; accept otherwise

E_{DFA} Example

New Deciders from Old: Equality Testing

$EQ_{\text{DFA}} = \{\langle D_1, D_2\rangle \mid D_1, D_2 \text{ are DFAs and } L(D_1) = L(D_2)\}$

Theorem: EQ_{DFA} is decidable

Proof: The following TM decides EQ_{DFA}

On input $\langle D_1, D_2\rangle$, where $\langle D_1, D_2\rangle$ are DFAs:

1. Construct a DFA D that recognizes the **symmetric difference** $L(D_1) \triangle L(D_2)$
2. Run the decider for E_{DFA} on $\langle D\rangle$ and return its output

Symmetric Difference

$$A \triangle B = \{w \mid w \in A \text{ or } w \in B \text{ but not both}\}$$

Meta-Computational Languages

$A_{\text{DFA}} = \{\langle D, w\rangle \mid \text{DFA } D \text{ accepts } w\}$

$A_{\text{TM}} = \{\langle M, w\rangle \mid \text{TM } M \text{ accepts } w\}$

$E_{\text{DFA}} = \{\langle D\rangle \mid \text{DFA } D \text{ recognizes the empty language } \emptyset\}$

$E_{\text{TM}} = \{\langle M\rangle \mid \text{TM } M \text{ recognizes the empty language } \emptyset\}$

$EQ_{\text{DFA}} = \{\langle D_1, D_2\rangle \mid D_1 \text{ and } D_2 \text{ are DFAs}, L(D_1) = L(D_2)\}$

$EQ_{\text{TM}} = \{\langle M_1, M_2\rangle \mid M_1 \text{ and } M_2 \text{ are TMs}, L(M_1) = L(M_2)\}$

The Universal Turing Machine

$A_{\text{TM}} = \{\langle M, w \rangle \mid M \text{ is a TM that accepts input } w\}$

Theorem: A_{TM} is Turing-recognizable

The following "universal TM" U recognizes A_{TM}

On input $\langle M, w \rangle$:

1. Simulate running M on input w
2. If M accepts, accept. If M rejects, reject.

Universal TM and A_{TM}

Why is the Universal TM not a decider for A_{TM}?

The following "universal TM" U recognizes A_{TM}

On input $\langle M, w \rangle$:
1. Simulate running M on input w
2. If M accepts, accept. If M rejects, reject.

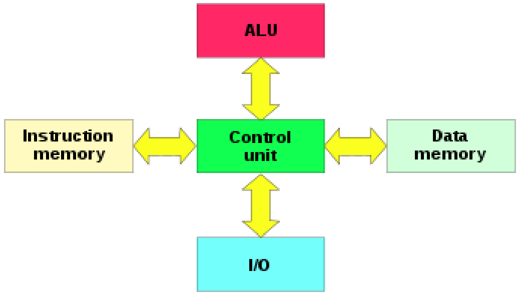
a) It may reject inputs $\langle M, w \rangle$ where M accepts w
b) It may accept inputs $\langle M, w \rangle$ where M rejects w
c) It may loop on inputs $\langle M, w \rangle$ where M loops on w
d) It may loop on inputs $\langle M, w \rangle$ where M rejects w

More on the Universal TM

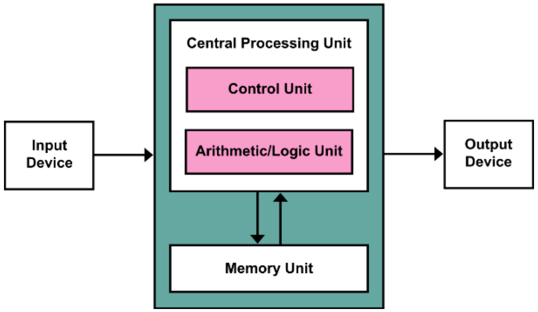
"It is possible to invent a single machine which can be used to compute any computable sequence. If this machine **U** is supplied with a tape on the beginning of which is written the S.D ["standard description"] of some computing machine **M**, then **U** will compute the same sequence as **M**."

- Turing, "On Computable Numbers…" 1936

- Foreshadowed general-purpose programmable computers
- No need for specialized hardware: Virtual machines as software



Harvard architecture:
Separate instruction and data pathways



von Neumann architecture:
Programs can be treated as data

Undecidability

A_{TM} is Turing-recognizable

…but it turns out A_{TM} (and $E_{\mathrm{TM}}, EQ_{\mathrm{TM}}$) is **undecidable**

i.e., computers cannot solve these problems no matter how much time they are given

Countability and Diagonalizaiton

What's your intuition?

Which of the following sets is the "biggest"?

a) The natural numbers: $\mathbb{N} = \{1, 2, 3, \ldots\}$

b) The even numbers: $E = \{2, 4, 6, \ldots\}$

c) The positive powers of 2: $POW2 = \{2, 4, 8, 16, \ldots\}$

d) They all have the same size

Set Theory Review

A function $f: A \rightarrow B$ is

- 1-to-1 (injective) if $f(a) \neq f(a')$ for all $a \neq a'$
- onto (surjective) if for all $b \in B$, there exists $a \in A$ such that $f(a) = b$
- a correspondence (bijective) if it is 1-to-1 and onto, i.e., every $b \in B$ has a unique $a \in A$ with $f(a) = b$

How can we compare sizes of infinite sets?

Definition: Two sets have the same size if there is a bijection between them

A set is countable if

- it is a finite set, or
- it has the same size as $\mathbb{N}$, the set of natural numbers

Examples of countable sets

- $\emptyset$
- $\{0,1\}$
- $\{0, 1, 2, \dots, 8675309\}$

- $E = \{2, 4, 6, 8, \dots\}$
- $SQUARES = \{1, 4, 9, 16, 25, \dots\}$
- $POW2 = \{2, 4, 8, 16, 32, \dots\}$

$$|E| = |SQUARES| = |POW2| = |\mathbb{N}|$$

How to show that $\mathbb{N} \times \mathbb{N}$ is countable?

$(1,1)$	$(2,1)$	$(3,1)$	$(4,1)$	$(5,1)$	$\ldots$
$(1,2)$	$(2,2)$	$(3,2)$	$(4,2)$	$(5,2)$	$\ldots$
$(1,3)$	$(2,3)$	$(3,3)$	$(4,3)$	$(5,3)$	$\ldots$
$(1,4)$	$(2,4)$	$(3,4)$	$(4,4)$	$(5,4)$	$\ldots$
$(1,5)$	$(2,5)$	$(3,5)$	$(4,5)$	$(5,5)$	
					$\ddots$

How to argue that a set S is countable

- Describe how to "list" the elements of S, usually in stages:

Ex: Stage 1) List all pairs (x, y) such that $x + y = 2$

Stage 2) List all pairs (x, y) such that $x + y = 3$

…

Stage n) List all pairs (x, y) such that $x + y = n + 1$

…

- Argue that every element of S appears in the list

Ex: Any $(x, y) \in \mathbb{N} \times \mathbb{N}$ will be listed in stage $x + y - 1$

- Define the bijection $f: \mathbb{N} \rightarrow S$ by $f(n) =$ the n'th element in this list (ignoring duplicates if needed)

Subsets of countable sets

If A and B are sets with $A \subseteq B$ (A is a subset of B), which of the following statements are true?

a) If A is countable, then B is countable

b) If B is countable, then A is countable

c) Both are true

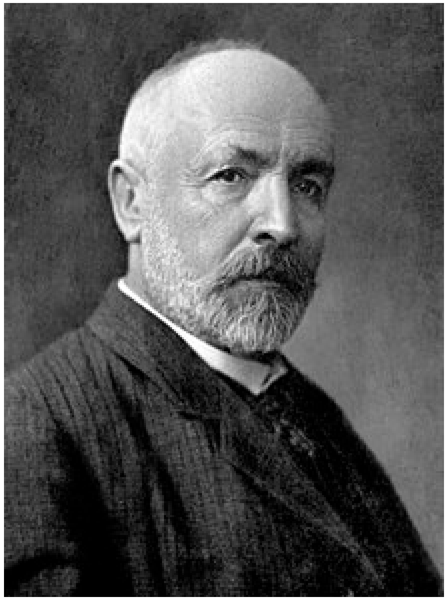
d) Neither is true

More examples of countable sets

- $\{0,1\}^*$
- $\{\langle M \rangle \mid M \text{ is a Turing machine}\}$
- $\mathbb{Q} = \{\text{rational numbers}\}$

So what *isn't* countable?

Cantor’s Diagonalization Method



Georg Cantor 1845-1918

- Invented set theory
- Defined countability, uncountability, cardinal and ordinal numbers, …

Some praise for his work:

“Scientific charlatan…renegade…corruptor of youth”
–L. Kronecker

“Set theory is wrong…utter nonsense…laughable”
–L. Wittgenstein

Uncountability of the reals

Theorem: The real interval $(0, 1)$ is uncountable.

Proof: Assume for the sake of contradiction it were countable, and let $f : \mathbb{N} \to (0,1)$ be a bijection

n	$f(n)$
1	$0 . d_1^1\ d_2^1\ d_3^1\ d_4^1\ d_5^1 \ldots$
2	$0 . d_1^2\ d_2^2\ d_3^2\ d_4^2\ d_5^2 \ldots$
3	$0 . d_1^3\ d_2^3\ d_3^3\ d_4^3\ d_5^3 \ldots$
4	$0 . d_1^4\ d_2^4\ d_3^4\ d_4^4\ d_5^4 \ldots$
5	$0 . d_1^5\ d_2^5\ d_3^5\ d_4^5\ d_5^5 \ldots$

Construct $b \in (0,1)$ which does not appear in this table
– contradiction!

$b = 0. b_1 b_2 b_3 \ldots$ where $b_i \neq d_i^i$ (digit i of $f(i)$)

Uncountability of the reals

A concrete example of the contradiction construction:

n	$f(n)$
1	0 . 8 6 7 5 3 0 9 …
2	0 . 1 4 1 5 9 2 6 …
3	0 . 7 1 8 2 8 1 8 …
4	0 . 4 4 4 4 4 4 4 …
5	0 . 1 3 3 7 1 3 3 …

Construct $b \in (0,1)$ which does not appear in this table – contradiction!

$b = 0.b_1 b_2 b_3 \ldots$ where $b_i \neq d_i^i$ (digit i of $f(i)$)

Diagonalization

This process of constructing a counterexample by "contradicting the diagonal" is called diagonalization

A general theorem about set sizes

Theorem: Let X be any set. Then the power set $P(X)$ does **not** have the same size as X.

Proof: Assume for the sake of contradiction that there is a bijection $f: X \to P(X)$

Goal: Construct a set $S \in P(X)$ that cannot be the output $f(x)$ for any $x \in X$

Diagonalization argument

Assume a correspondence $f: X \to P(X)$

x					
x_1					
x_2					
x_3					
x_4					
$\vdots$					

Diagonalization argument

Assume a correspondence $f: X \rightarrow P(X)$

x	$x_1 \in f(x)?$	$x_2 \in f(x)?$	$x_3 \in f(x)?$	$x_4 \in f(x)?$	…
x_1	Y	N	Y	Y	
x_2	N	N	Y	Y	
x_3	Y	Y	Y	N	
x_4	N	N	Y	N	
$\vdots$					$\ddots$

Define S by flipping the diagonal:

Put $\quad x_i \in S \quad \Longleftrightarrow \quad x_i \notin f(x_i)$

Example

Let $X = \{1, 2, 3\}$, $P(X) = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}$

Ex. $f(1) = \{1, 2\}$, $f(2) = \emptyset$, $f(3) = \{2\}$

x	$1 \in f(x)$?	$2 \in f(x)$?	$3 \in f(x)$?
1			
2			
3			

Construct $S =$

A general theorem about set sizes

Theorem: Let X be any set. Then the power set $P(X)$ does **not** have the same size as X.

Proof: Assume for the sake of contradiction that there is a bijection $f: X \to P(X)$

Construct a set $S \in P(X)$ that cannot be the output $f(x)$ for any $x \in X$:

$$S = \{x \in X \mid x \notin f(x)\}$$

If $S = f(y)$ for some $y \in X$,

then $y \in S$ if and only if $y \notin S$