BU CS 332 – Theory of Computation

Lecture 14:

- More on Diagonalization
- Undecidability

Reading:

Sipser Ch 4.2

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How can we compare sizes of infinite sets?

Definition: Two sets have the same size if there is a bijection between them

A set is countable if

- it is a finite set, or
- it has the same size as $\mathbb{N}$, the set of natural numbers

Uncountability of the reals

Theorem: The real interval $(0, 1)$ is uncountable.

Proof: Assume for the sake of contradiction it were countable, and let $f: \mathbb{N} \to (0,1)$ be a bijection

n	$f(n)$
1	$0\,.\,d_1^1\ d_2^1\ d_3^1\ d_4^1\ d_5^1 \ldots$
2	$0\,.\,d_1^2\ d_2^2\ d_3^2\ d_4^2\ d_5^2 \ldots$
3	$0\,.\,d_1^3\ d_2^3\ d_3^3\ d_4^3\ d_5^3 \ldots$
4	$0\,.\,d_1^4\ d_2^4\ d_3^4\ d_4^4\ d_5^4 \ldots$
5	$0\,.\,d_1^5\ d_2^5\ d_3^5\ d_4^5\ d_5^5 \ldots$

Construct $b \in (0,1)$ which does not appear in this table:

$b = 0. b_1 b_2 b_3 \ldots$ where $b_n \neq d_n^n$ (digit n of $f(n)$)

There is no n for which $f(n) = b$, which contradicts the assumption that f is onto

Uncountability of the reals

A concrete example of the contradiction construction:

n	$f(n)$
1	$0.8675309\ldots$
2	$0.1415926\ldots$
3	$0.7182818\ldots$
4	$0.4444444\ldots$
5	$0.1337133\ldots$

Construct $b \in (0,1)$ which does not appear in this table

$b = 0.b_1 b_2 b_3 \ldots$ where $b_n \neq d_n^n$ (digit n of $f(n)$)

Diagonalization

This process of constructing a counterexample by "contradicting the diagonal" is called diagonalization

Structure of a diagonalization proof

Say you want to show that a set T is uncountable

1) Assume, for the sake of contradiction, that T is countable with bijection $f : \mathbb{N} \to T$
2) "Flip the diagonal" to construct an element $b \in T$ such that $f(n) \neq b$ for every n

   Ex: Let $b = 0.b_1 b_2 b_3 \ldots$ where $b_n \neq d_n^n$
   (where d_n^n is digit n of $f(n)$)

3) Conclude that f is not onto, contradicting assumption that f is a bijection

A general theorem about set sizes

Theorem: Let X be any set. Then the power set $P(X)$ does **not** have the same size as X.

Proof: Assume for the sake of contradiction that there is a bijection $f: X \to P(X)$

Goal: Construct a set $S \in P(X)$ that cannot be the output $f(x)$ for any $x \in X$

Diagonalization argument

Assume a correspondence $f: X \to P(X)$

x					
x_1					
x_2					
x_3					
x_4					
$\vdots$					

Diagonalization argument

Assume a correspondence $f: X \to P(X)$

x	$x_1 \in f(x)$?	$x_2 \in f(x)$?	$x_3 \in f(x)$?	$x_4 \in f(x)$?	…
x_1	Y	N	Y	Y	
x_2	N	N	Y	Y	
x_3	Y	Y	Y	N	
x_4	N	N	Y	N	
$\vdots$					$\ddots$

Define S by flipping the diagonal:

Put $\quad x_n \in S \quad \Leftrightarrow \quad x_n \notin f(x_n)$

Example

Let $X = \{1, 2, 3\}$, $P(X) = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}$

Ex. $f(1) = \{1, 2\}$, $f(2) = \emptyset$, $f(3) = \{2\}$

x	$1 \in f(x)$?	$2 \in f(x)$?	$3 \in f(x)$?
1			
2			
3			

Construct $S =$ a) $\{1\}$ c) $\{2, 3\}$

b) $\{1, 2, 3\}$ d) $\emptyset$

A general theorem about set sizes

Theorem: Let X be any set. Then the power set $P(X)$ does **not** have the same size as X.

Proof: Assume for the sake of contradiction that there is a bijection $f: X \to P(X)$

Construct a set $S \in P(X)$ that cannot be the output $f(x)$ for any $x \in X$:

$$S = \{x \in X \mid x \notin f(x)\}$$

If $S = f(y)$ for some $y \in X$,

then $y \in S$ if and only if $y \notin S$

Undecidable Languages

Undecidability / Unrecognizability

Definition: A language L is undecidable if there is no TM deciding L

Definition: A language L is unrecognizable if there is no TM recognizing L

An existential proof

Theorem: There exists an undecidable language over $\{0, 1\}$

Proof:

Set of all encodings of TM deciders: $X \subseteq \{0, 1\}^*$

Set of all languages over $\{0, 1\}$:

a) $\{0, 1\}$
b) $\{0, 1\}^*$
c) $P(\{0, 1\}^*)$: The set of all subsets of $\{0, 1\}^*$
d) $P(P(\{0, 1\}^*))$: The set of all subsets of the set of all subsets of $\{0, 1\}^*$

An existential proof

Theorem: There exists an undecidable language over $\{0, 1\}$

Proof:

Set of all encodings of TM deciders: $X \subseteq \{0, 1\}^*$

Set of all languages over $\{0, 1\}$: $P(\{0, 1\}^*)$

There are more languages than there are TM deciders!

$\Rightarrow$ There must be an undecidable language

An existential proof

Theorem: There exists an unrecognizable language over $\{0,1\}$

Proof:

Set of all encodings of TMs: $X \subseteq \{0,1\}^*$

Set of all languages over $\{0,1\}$: $P(\{0,1\}^*)$

There are more languages than there are TM deciders!

$\Rightarrow$ There must be an unrecognizable language

“Almost all” languages are undecidable

So how about we find one?

An explicit undecidable language

TM M					
M_1					
M_2					
M_3					
M_4					
$\vdots$					

Why is it possible to enumerate all TMs like this?

a) The set of all TM deciders is finite
b) The set of all TM deciders is countably infinite
c) The set of all TM deciders is uncountable

An explicit undecidable language

TM M	$M(\langle M_1 \rangle)$?	$M(\langle M_2 \rangle)$?	$M(\langle M_3 \rangle)$?	$M(\langle M_4 \rangle)$?		$D(\langle D \rangle)$?
M_1	Y	N	Y	Y	…	
M_2	N	N	Y	Y		
M_3	Y	Y	Y	N		
M_4	N	N	Y	N		
$\vdots$					$\ddots$	
D						

$UD = \{\langle M \rangle \mid M \text{ is a TM that does not accept on input } \langle M \rangle\}$

Suppose D decides UD

An explicit undecidable language

Theorem: $UD = \{\langle M \rangle \mid M \text{ is a TM that does not accept on input } \langle M \rangle\}$ is undecidable

Proof: Suppose for contradiction, that TM D decides UD

A more useful undecidable language

$A_{\mathrm{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM that accepts input } w\}$

Theorem: A_{TM} is undecidable

Proof: Assume for the sake of contradiction that TM H decides A_{TM}:

$$H(\langle M, w\rangle) = \begin{cases} \text{accept} & \text{if } M \text{ accepts } w \\ \text{reject} & \text{if } M \text{ does not accept } w \end{cases}$$

Idea: Show that H can be used to decide the (undecidable) language UD -- a contradiction.

A more useful undecidable language

$A_{\text{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM that accepts input } w\}$

Proof (continued):

Suppose, for contradiction, that H decides A_{TM}

Consider the following TM D:

"On input $\langle M \rangle$ where M is a TM:

1. Run H on input $\langle M, \langle M \rangle \rangle$
2. If H accepts, reject. If H rejects, accept."

Claim: D decides $UD = \{\langle M \rangle \mid \text{TM } M \text{ does not accept } \langle M \rangle\}$

…but this language is undecidable

Unrecognizable Languages

Theorem: A language L is decidable if and only if L and $\overline{L}$ are both Turing-recognizable.

Proof:

Unrecognizable Languages

Theorem: A language L is decidable if and only if L and $\overline{L}$ are both Turing-recognizable.

Proof:

Classes of Languages