BU CS 332 – Theory of Computation

Lecture 15:

- Review mid-semester feedback
- Reductions

Reading:

Sipser Ch 5.1

Mark Bun

March 15, 2021

What helps you learn best?

- Lectures in general (13)
- In-class examples / walkthroughs (11)
- Interaction in lecture, polls (7)
- Discussion sections in general (7)
- Gradescope check-ins (5)
- Homework – useful, appropriate length/difficulty (5)
- Automata Tutor, TM simulator (4)
- Office hours (4)
- Course organization, perspective (3)
- Annotated slides (3)
- Piazza use (3)
- Homework feedback
- Reading

What hinders your learning?

- Remote format, COVID / Zoom fatigue (7)
- Course difficulty, recent increase in difficulty (4)
- Proofs, proof assignments on homework (2)
- Too theoretical / knowledge of concepts but not how to use them (2)
- Homework too long, too difficult (2)
- Automata Tutor / Morphett
- Turing machines
- Starting homework late
- Vague answers in office hours
- Transferring lecture knowledge to homework
- Delay on homework feedback
- Sipser book
- Difficulty finding collaborators
- Gradescope check-ins
- Instructor mistakes
- Weekly (vs. less frequent) assignments
- Instructor handwriting
- Course pace too fast
- Can't turn in late work

Suggestions for course improvement

- More office hours (3)
- More examples (3)
- Recommendations for what's expected on HW solutions (3)
- More polls, interaction (2)
- Better / more visible handwriting, or type on slides (2)
- Faster turnaround on grades (2)
- +12 hours on HW submissions (2)
- Supplementary readings / videos (2)
- Shorter breakouts in discussion
- Releasing homework earlier
- Guidance on how to prove things
- Homeworks build up from easier to harder questions
- Homework solutions
- Tutoring sessions
- Old exam questions during discussion
- More ungraded practice
- Less difficult homework
- Slower lectures
- Free A's

Clarity of expectations

- Seems mostly clear
- Reminder of resources to take advantage of:

  Sipser textbook

  Lectures (slides, recordings, Gradescope check-ins)

  Discussions (in-class meetings, solution recording, posted slides)

  Homework feedback, posted solutions

  Office hours

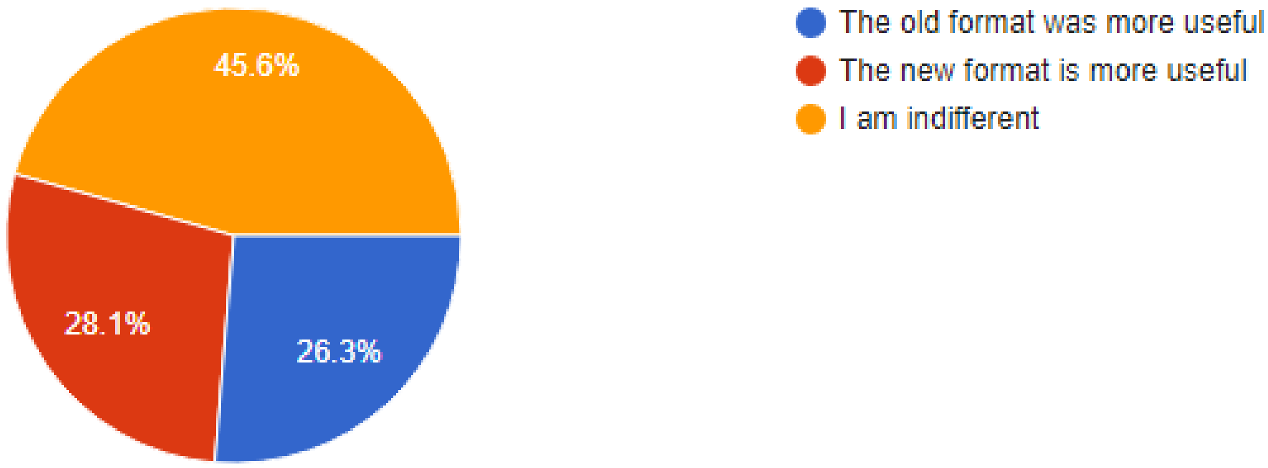
  Piazza

- See Lecture 1, Slides 13-17 for more advice

Suggestions for self-improvement

- Keep up with readings (24)
  "I believe reading the textbook is more helpful that students realize. I have been reading the textbook inconsistently and I find the weeks that I do the reading, I can better understand the lecture material."
- Time management (10)
- Review lecture / discussion materials (8)
- Attend more office hours (5)
- Participate in class more actively (5)
- Attend more discussions (2)
- Do example problems in Sipser (2)
- Find collaborators
- Remember to do Gradescope check-ins
- Participate on Piazza

Discussion format / feedback



- Nadya’s awesome (9)
- Breakout rooms can be awkward / not useful (3)
- New format is more engaging, allow for solving more problems (2)
- Discussion problems too easy

Proposed Course Modifications

- Poll for more office hours, tutoring
- Discussions
  - Keeping new format (class time for breakout rooms, solution video after)
  - Nadya will kick start discussion in quieter groups
- Homework more approachable and useful
  - More explicit guidance on components of a complete solution (See also lecture slides where I try to do this)
  - Gradient from easier (mechanical) to harder (creative) questions

Other questions / concerns

- Is the final cumulative?

  Yes, with an emphasis on last third of material

- Is there a curve?

  Can expect grade increases for scores especially in the 50-75% range

- Specific concerns about test grades (especially relative to HW)

  Come talk to us about this. Office hours can be an awkward time, so schedule an appointment

Undecidability and Reductions

Undecidability / Unrecognizability

Definition: A language L is undecidable if there is no TM deciding L

Definition: A language L is unrecognizable if there is no TM recognizing L

Last time: Two explicit undecidable languages

$UD = \{\langle M \rangle \mid M \text{ is a TM that does not accept on input } \langle M \rangle\}$

- Shown directly by diagonalization

$A_{\text{TM}} = \{\langle M, w \rangle \mid M \text{ is a TM that accepts input } w\}$

- "Reduction" from the undecidability of UD

Scientists vs. Engineers

A computer scientist and an engineer are stranded on a desert island. They find two palm trees with one coconut on each. The engineer climbs a tree, picks a coconut and eats.

The computer scientist climbs the second tree, picks a coconut, climbs down, climbs up the first tree and places it there, declaring success.

"Now we've reduced the problem to one we've already solved."

(Please laugh)

Reductions

A reduction from problem A to problem B is an algorithm for problem A which uses an algorithm for problem B as a subroutine

If such a reduction exists, we say "A reduces to B"

Reductions

A reduction from problem A to problem B is an algorithm for problem A which uses an algorithm for problem B as a subroutine

If such a reduction exists, we say "A reduces to B"

If A reduces to B, and B is decidable, what can we say about A?

a) A is decidable

b) A is undecidable

c) A might be either decidable or undecidable

Two uses of reductions

Positive uses: If A reduces to B and B is decidable, then A is also decidable

$EQ_{\mathrm{DFA}} = \{\langle D_1, D_2 \rangle \mid D_1, D_2 \text{ are DFAs and } L(D_1) = L(D_2)\}$

Theorem: EQ_{DFA} is decidable

Proof: The following TM decides EQ_{DFA}

On input $\langle D_1, D_2 \rangle$, where $\langle D_1, D_2 \rangle$ are DFAs:

1. Construct a DFA D that recognizes the symmetric difference $L(D_1) \triangle L(D_2)$
2. Run the decider for E_{DFA} on $\langle D \rangle$ and return its output

Two uses of reductions

Negative uses: If A reduces to B and A is undecidable, then B is also undecidable

$A_{\text{TM}} = \{\langle M, w \rangle \mid M \text{ is a TM that accepts input } w\}$

Suppose H decides A_{TM}

Consider the following TM D.

On input $\langle M \rangle$ where M is a TM:

1. Run H on input $\langle M, \langle M \rangle \rangle$
2. If H accepts, reject. If H rejects, accept.

Claim: D decides

$$UD = \{\langle M \rangle \mid M \text{ is a TM that does not accept input } \langle M \rangle\}$$

Two uses of reductions

Negative uses: If A reduces to B and A is undecidable, then B is also undecidable

Template for undecidability proof by reduction:

1. Suppose to the contrary that B is decidable
2. Using a decider for B as a subroutine, construct an algorithm deciding A
3. But A is undecidable. Contradiction!

Halting Problem

Computational problem: Given a program (TM) and input w, does that program halt (either accept or reject) on input w?

Formulation as a language:

$$HALT_{\mathrm{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM that halts on input } w\}$$

Ex. M = "On input x (a natural number in binary):

For each $y = 1, 2, 3, \ldots$:

If $y^2 = x$, accept. Else, continue."

Is $\langle M, 101\rangle \in HALT_{\mathrm{TM}}$?

a) Yes, because M accepts on input 101
b) Yes, because M rejects on input 101
c) No, because M rejects on input 101
d) No, because M loops on input 101

Halting Problem

Computational problem: Given a program (TM) and input w, does that program halt on input w?

Formulation as a language:

$$HALT_{\mathrm{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM that halts on input } w\}$$

Ex. M = "On input x (a natural number in binary):

For each $y = 1, 2, 3, \ldots$:

If $y^2 = x$, accept. Else, continue."

M' = "On input x (a natural number in binary):

For each $y = 1, 2, 3, \ldots, x$:

If $y^2 = x$, accept. Else, continue.

Reject."

Halting Problem

$$HALT_{\mathrm{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM that halts on input } w\}$$

Theorem: $HALT_{\mathrm{TM}}$ is undecidable

Proof: Suppose for contradiction that there exists a decider H for $HALT_{\mathrm{TM}}$. We construct a decider for A_{TM} as follows:

On input $\langle M, w\rangle$:

1. Run H on input $\langle M, w\rangle$
2. If H rejects, reject
3. If H accepts, run M on w
4. If M accepts, accept

   Otherwise, reject.

This is a reduction from A_{TM} to $HALT_{\mathrm{TM}}$

Halting Problem

Computational problem: Given a program (TM) and input w, does that program halt on input w?

- A central problem in formal verification
- Dealing with undecidability in practice:
  - Use heuristics that are correct on most real instances, but may be wrong or loop forever on others
  - Restrict to a "non-Turing-complete" subclass of programs for which halting is decidable
  - Use a programming language that lets a programmer specify hints (e.g., loop invariants) that can be compiled into a formal proof of halting

Empty language testing for TMs

$$E_{\text{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$$

Theorem: E_{TM} is undecidable

Proof: Suppose for contradiction that there exists a decider R for E_{TM}. We construct a decider for A_{TM} as follows:

On input $\langle M, w \rangle$:

1. Run R on input ???

This is a reduction from A_{TM} to E_{TM}

Empty language testing for TMs

$$E_{\text{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$$

Theorem: E_{TM} is undecidable

Proof: Suppose for contradiction that there exists a decider R for E_{TM}. We construct a decider for A_{TM} as follows:

On input $\langle M, w \rangle$:

1. Construct a TM N as follows:

2. Run R on input $\langle N \rangle$

3. If R , accept. Otherwise, reject

What do we want out of machine N?

a) $L(N)$ is empty iff M accepts w

b) $L(N)$ is non-empty iff M accepts w

c) $L(M)$ is empty iff N accepts w

d) $L(M)$ is non-empty iff N accepts w

This is a reduction from A_{TM} to E_{TM}

Empty language testing for TMs

$$E_{\text{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$$

Theorem: E_{TM} is undecidable

Proof: Suppose for contradiction that there exists a decider R for E_{TM}. We construct a decider for A_{TM} as follows:

On input $\langle M, w \rangle$:

1. Construct a TM N as follows:

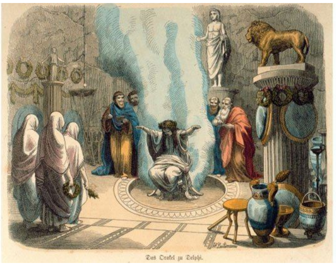
   "On input x:

   Run M on w and output the result."

2. Run R on input $\langle N \rangle$
3. If R rejects, accept. Otherwise, reject

This is a reduction from A_{TM} to E_{TM}

Interlude: Formalizing Reductions (Sipser 6.3)



Informally: A reduces to B if a decider for B can be used to construct a decider for A

One way to formalize:

- An *oracle* for language B is a device that can answer questions "Is $w \in B$?"
- An *oracle TM* M^B is a TM that can query an oracle for B in one computational step

A is Turing-reducible to B (written $A \leq_T B$) if there is an oracle TM M^B deciding A

Equality Testing for TMs

$$EQ_{\text{TM}} = \{\langle M_1, M_2\rangle \mid M_1, M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$$

Theorem: EQ_{TM} is undecidable

Proof: Suppose for contradiction that there exists a decider R for EQ_{TM}. We construct a decider for A_{TM} as follows:

On input $\langle M, w\rangle$:

1. Construct TMs N_1, N_2 as follows:

   N_1 = N_2 =

2. Run R on input $\langle N_1, N_2\rangle$
3. If R accepts, accept. Otherwise, reject.

This is a reduction from A_{TM} to EQ_{TM}

Regular language testing for TMs

$$REG_{\text{TM}} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) \text{ is regular}\}$$

Theorem: REG_{TM} is undecidable

Proof: Suppose for contradiction that there exists a decider R for REG_{TM}. We construct a decider for A_{TM} as follows:

On input $\langle M, w\rangle$:

1. Construct a TM N as follows:

2. Run R on input $\langle N\rangle$

3. If R accepts, accept. Otherwise, reject

This is a reduction from A_{TM} to REG_{TM}

Regular language testing for TMs

$$REG_{\text{TM}} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) \text{ is regular}\}$$

Theorem: REG_{TM} is undecidable

Proof: Suppose for contradiction that there exists a decider R for REG_{TM}. We construct a decider for A_{TM} as follows:

On input $\langle M, w\rangle$:

1. Construct a TM N as follows:

   N = "On input x,

   1. If $x \in \{0^n1^n \mid n \geq 0\}$, accept
   2. Run TM M on input w
   3. If M accepts, accept. Otherwise, reject."

2. Run R on input $\langle N\rangle$
3. If R accepts, accept. Otherwise, reject

This is a reduction from A_{TM} to REG_{TM}