BU CS 332 – Theory of Computation

Lecture 15:

• Review mid-semester feedback
• Reductions

Reading:
Sipser Ch 5.1

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What helps you learn best?

• Lectures in general (13)
• In-class examples / walkthroughs (11)
• Interaction in lecture, polls (7)
• Discussion sections in general (7)
• Gradescope check-ins (5)
• Homework – useful, appropriate length/difficulty (5)
• Automata Tutor, TM simulator (4)
• Office hours (4)
• Course organization, perspective (3)
• Annotated slides (3)
• Piazza use (3)
• Homework feedback
• Reading
What hinders your learning?

- Remote format, COVID / Zoom fatigue (7)
- Course difficulty, recent increase in difficulty (4)
- Proofs, proof assignments on homework (2)
- Too theoretical / knowledge of concepts but not how to use them (2)
- Homework too long, too difficult (2)
- Automata Tutor / Morphett
- Turing machines
- Starting homework late
- Vague answers in office hours
- Transferring lecture knowledge to homework
- Delay on homework feedback
- Sipser book
- Difficulty finding collaborators
- Gradescope check-ins
- Instructor mistakes
- Weekly (vs. less frequent) assignments
- Instructor handwriting
- Course pace too fast
- Can’t turn in late work
Suggestions for course improvement

• More office hours (3)
• More examples (3)
• Recommendations for what’s expected on HW solutions (3)
• More polls, interaction (2)
• Better / more visible handwriting, or type on slides (2)
• Faster turnaround on grades (2)
• +12 hours on HW submissions (2)
• Supplementary readings / videos (2)
• Shorter breakouts in discussion
• Releasing homework earlier
• Guidance on how to prove things
• Homeworks build up from easier to harder questions
• Homework solutions
• Tutoring sessions
• Old exam questions during discussion
• More ungraded practice
• Less difficult homework
• Slower lectures
• Free A’s
Clarity of expectations

• Seems mostly clear

• Reminder of resources to take advantage of:
  Sipser textbook
  Lectures (slides, recordings, Gradescope check-ins)
  Discussions (in-class meetings, solution recording, posted slides)
  Homework feedback, posted solutions
  Office hours
  Piazza

• See Lecture 1, Slides 13-17 for more advice
Suggestions for self-improvement

• Keep up with readings (24)
  “I believe reading the textbook is more helpful that students realize. I have been reading the textbook inconsistently and I find the weeks that I do the reading, I can better understand the lecture material.”

• Time management (10)
• Review lecture / discussion materials (8)
• Attend more office hours (5)
• Participate in class more actively (5)
• Attend more discussions (2)
• Do example problems in Sipser (2)
• Find collaborators
• Remember to do Gradescope check-ins
• Participate on Piazza
Discussion format / feedback

- Nadya’s awesome (9)
- Breakout rooms can be awkward / not useful (3)
- New format is more engaging, allow for solving more problems (2)
- Discussion problems too easy
Proposed Course Modifications

• Poll for more office hours, tutoring

• Discussions
  - Keeping new format (class time for breakout rooms, solution video after)
  - Nadya will kick start discussion in quieter groups

• Homework more approachable and useful
  - More explicit guidance on components of a complete solution (See also lecture slides where I try to do this)
  - Gradient from easier (mechanical) to harder (creative) questions
Other questions / concerns

• Is the final cumulative?
  Yes, with an emphasis on last third of material

• Is there a curve?
  Can expect grade increases for scores especially in the 50-75% range

• Specific concerns about test grades (especially relative to HW)
  Come talk to us about this. Office hours can be an awkward time, so schedule an appointment
Undecidability and Reductions
Undecidability / Unrecognizability

**Definition:** A language $L$ is undecidable if there is no TM deciding $L$

**Definition:** A language $L$ is unrecognizable if there is no TM recognizing $L$
Last time: Two explicit undecidable languages

\[ UD = \{ \langle M \rangle \mid M \text{ is a TM that does not accept on input } \langle M \rangle \} \]

- Shown directly by diagonalization

\[ A_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM that accepts input } w \} \]

- “Reduction” from the undecidability of \( UD \)
Scientists vs. Engineers

A computer scientist and an engineer are stranded on a desert island. They find two palm trees with one coconut on each. The engineer climbs a tree, picks a coconut and eats.

The computer scientist climbs the second tree, picks a coconut, climbs down, climbs up the first tree and places it there, declaring success.

“Now we’ve reduced the problem to one we’ve already solved.” (Please laugh)
Reductions

A reduction from problem $A$ to problem $B$ is an algorithm for problem $A$ which uses an algorithm for problem $B$ as a subroutine.

If such a reduction exists, we say “$A$ reduces to $B$”
Reductions

A reduction from problem $A$ to problem $B$ is an algorithm for problem $A$ which uses an algorithm for problem $B$ as a subroutine.

If such a reduction exists, we say “$A$ reduces to $B$”.

If $A$ reduces to $B$, and $B$ is decidable, what can we say about $A$?

a) $A$ is decidable
b) $A$ is undecidable
c) $A$ might be either decidable or undecidable
Two uses of reductions

Positive uses: If $A$ reduces to $B$ and $B$ is decidable, then $A$ is also decidable

$EQ_{DFA} = \{\langle D_1, D_2 \rangle \mid D_1, D_2 \text{ are DFAs and } L(D_1) = L(D_2)\}$

Theorem: $EQ_{DFA}$ is decidable

Proof: The following TM decides $EQ_{DFA}$

On input $\langle D_1, D_2 \rangle$, where $\langle D_1, D_2 \rangle$ are DFAs:

1. Construct a DFA $D$ that recognizes the symmetric difference $L(D_1) \triangle L(D_2)$

2. Run the decider for $E_{DFA}$ on $\langle D \rangle$ and return its output
Two uses of reductions

**Negative uses:** If \( A \) reduces to \( B \) and \( A \) is undecidable, then \( B \) is also undecidable

\[
A_{TM} = \{\langle M, w \rangle \mid M \text{ is a TM that accepts input } w\}
\]
Suppose \( H \) decides \( A_{TM} \)

Consider the following TM \( D \).

On input \( \langle M \rangle \) where \( M \) is a TM:
1. Run \( H \) on input \( \langle M, \langle M \rangle \rangle \)
2. If \( H \) accepts, reject. If \( H \) rejects, accept.

**Claim:** \( D \) decides
\[
UD = \{\langle M \rangle \mid M \text{ is a TM that does not accept input } \langle M \rangle\}
\]
Two uses of reductions

Negative uses: If $A$ reduces to $B$ and $A$ is undecidable, then $B$ is also undecidable

Template for undecidability proof by reduction:

1. Suppose to the contrary that $B$ is decidable
2. Using a decider for $B$ as a subroutine, construct an algorithm deciding $A$
3. But $A$ is undecidable. Contradiction!
Halting Problem

Computational problem: Given a program (TM) and input \( w \), does that program halt (either accept or reject) on input \( w \)?

Formulation as a language:
\[ \text{HALT}_{TM} = \{ \langle M, w \rangle | M \text{ is a TM that halts on input } w \} \]

Ex. \( M = \) “On input \( x \) (a natural number in binary):
    For each \( y = 1, 2, 3, \ldots \):
        If \( y^2 = x \), accept. Else, continue.”

Is \( \langle M, 101 \rangle \in \text{HALT}_{TM} \)?

a) Yes, because \( M \) accepts on input 101
b) Yes, because \( M \) rejects on input 101
c) No, because \( M \) rejects on input 101
d) No, because \( M \) loops on input 101
Halting Problem

Computational problem: Given a program (TM) and input \( w \), does that program halt on input \( w \)?

Formulation as a language:
\[
HALT_{TM} = \{ \langle M, w \rangle | M \text{ is a TM that halts on input } w \}\]

Ex. \( M = \) “On input \( x \) (a natural number in binary):
   For each \( y = 1, 2, 3, \ldots \):
      If \( y^2 = x \), accept. Else, continue.”

\( M' = \) “On input \( x \) (a natural number in binary):
   For each \( y = 1, 2, 3, \ldots, x \):
      If \( y^2 = x \), accept. Else, continue.
      Reject.”
Halting Problem

\[ \text{HALT}_{\text{TM}} = \{ \langle M, w \rangle \mid M \text{ is a TM that halts on input } w \} \]

Theorem: \( \text{HALT}_{\text{TM}} \) is undecidable

Proof: Suppose for contradiction that there exists a decider \( H \) for \( \text{HALT}_{\text{TM}} \). We construct a decider for \( A_{\text{TM}} \) as follows:

On input \( \langle M, w \rangle \):
1. Run \( H \) on input \( \langle M, w \rangle \)
2. If \( H \) rejects, reject
3. If \( H \) accepts, run \( M \) on \( w \)
4. If \( M \) accepts, accept
   Otherwise, reject.

This is a reduction from \( A_{\text{TM}} \) to \( \text{HALT}_{\text{TM}} \)
Halting Problem

Computational problem: Given a program (TM) and input $w$, does that program halt on input $w$?

- A central problem in formal verification

- Dealing with undecidability in practice:
  - Use heuristics that are correct on most real instances, but may be wrong or loop forever on others
  - Restrict to a “non-Turing-complete” subclass of programs for which halting is decidable
  - Use a programming language that lets a programmer specify hints (e.g., loop invariants) that can be compiled into a formal proof of halting
Empty language testing for TMs

\[ E_{TM} = \{ \langle M \rangle | M \text{ is a TM and } L(M) = \emptyset \} \]

**Theorem:** \( E_{TM} \) is undecidable

**Proof:** Suppose for contradiction that there exists a decider \( R \) for \( E_{TM} \). We construct a decider for \( A_{TM} \) as follows:

On input \( \langle M, w \rangle \):

1. Run \( R \) on input ???

This is a reduction from \( A_{TM} \) to \( E_{TM} \)
Empty language testing for TMs

\[ E_{\text{TM}} = \{ \langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset \} \]

Theorem: \( E_{\text{TM}} \) is undecidable

Proof: Suppose for contradiction that there exists a decider \( R \) for \( E_{\text{TM}} \). We construct a decider for \( A_{\text{TM}} \) as follows:

On input \( \langle M, w \rangle \):

1. Construct a TM \( N \) as follows:
2. Run \( R \) on input \( \langle N \rangle \)
3. If \( R \) accepts, accept. Otherwise, reject

What do we want out of machine \( N \)?

a) \( L(N) \) is empty iff \( M \) accepts \( w \)
b) \( L(N) \) is non-empty iff \( M \) accepts \( w \)
c) \( L(M) \) is empty iff \( N \) accepts \( w \)
d) \( L(M) \) is non-empty iff \( N \) accepts \( w \)

This is a reduction from \( A_{\text{TM}} \) to \( E_{\text{TM}} \).
Empty language testing for TMs

\[ E_{TM} = \{ \langle M \rangle | M \text{ is a TM and } L(M) = \emptyset \} \]

**Theorem:** \( E_{TM} \) is undecidable

**Proof:** Suppose for contradiction that there exists a decider \( R \) for \( E_{TM} \). We construct a decider for \( A_{TM} \) as follows:

On input \( \langle M, w \rangle \):

1. Construct a TM \( N \) as follows:
   “On input \( x \):
   Run \( M \) on \( w \) and output the result.”
2. Run \( R \) on input \( \langle N \rangle \)
3. If \( R \) rejects, accept. Otherwise, reject

This is a reduction from \( A_{TM} \) to \( E_{TM} \)
Interlude: Formalizing Reductions (Sipser 6.3)

Informally: $A$ reduces to $B$ if a decider for $B$ can be used to construct a decider for $A$

One way to formalize:

- An *oracle* for language $B$ is a device that can answer questions “Is $w \in B$?”
- An *oracle TM* $M^B$ is a TM that can query an oracle for $B$ in one computational step

$A$ is Turing-reducible to $B$ (written $A \leq_T B$) if there is an oracle TM $M^B$ deciding $A$
Equality Testing for TMs

\[ EQ_{TM} = \{ \langle M_1, M_2 \rangle \mid M_1, M_2 \text{ are TMs and } L(M_1) = L(M_2) \} \]

**Theorem:** \( EQ_{TM} \) is undecidable

**Proof:** Suppose for contradiction that there exists a decider \( R \) for \( EQ_{TM} \). We construct a decider for \( A_{TM} \) as follows:

**On input \( \langle M, w \rangle \):**
1. Construct TMs \( N_1, N_2 \) as follows:
   \[ N_1 = \quad N_2 = \]
2. Run \( R \) on input \( \langle N_1, N_2 \rangle \)
3. If \( R \) accepts, *accept*. Otherwise, *reject*.

This is a reduction from \( A_{TM} \) to \( EQ_{TM} \)
Regular language testing for TMs

$$REG_{TM} = \{ \langle M \rangle | M \text{ is a TM and } L(M) \text{ is regular} \}$$

**Theorem:** $REG_{TM}$ is undecidable

**Proof:** Suppose for contradiction that there exists a decider $R$ for $REG_{TM}$. We construct a decider for $A_{TM}$ as follows:

On input $\langle M, w \rangle$:

1. Construct a TM $N$ as follows:
   2. Run $R$ on input $\langle N \rangle$
   3. If $R$ accepts, accept. Otherwise, reject

This is a reduction from $A_{TM}$ to $REG_{TM}$
Regular language testing for TMs

\[ \text{REG}_{\text{TM}} = \{ \langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is regular} \} \]

**Theorem:** \( \text{REG}_{\text{TM}} \) is undecidable

**Proof:** Suppose for contradiction that there exists a decider \( R \) for \( \text{REG}_{\text{TM}} \). We construct a decider for \( A_{\text{TM}} \) as follows:

On input \( \langle M, w \rangle \):

1. Construct a TM \( N \) as follows:
   \[
   N = \text{"On input } x, \text{ accept if } x \in \{0^n1^n \mid n \geq 0\}, \text{ run } M \text{ on input } w, \text{ accept if } M \text{ accepts, reject."
   }
   
2. Run \( R \) on input \( \langle N \rangle \)
3. If \( R \) accepts, accept. Otherwise, reject.

This is a reduction from \( A_{\text{TM}} \) to \( \text{REG}_{\text{TM}} \)