Lecture 17: Mapping Reductions

Reading: Sipser Ch 5.3

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Reductions

A **reduction** from problem $A$ to problem $B$ is an algorithm for problem $A$ which uses an algorithm for problem $B$ as a subroutine.

If such a reduction exists, we say “$A$ reduces to $B$”.

**Positive uses:** If $A$ reduces to $B$ and $B$ is decidable, then $A$ is also decidable.

Ex. $E_{DFA}$ is decidable $\Rightarrow E_{Q_{DFA}}$ is decidable

**Negative uses:** If $A$ reduces to $B$ and $A$ is undecidable, then $B$ is also undecidable.

Ex. $E_{TM}$ is undecidable $\Rightarrow E_{Q_{TM}}$ is undecidable.
What’s wrong with the following “proof”? 

Bogus “Theorem”: \( A_{TM} \) is not Turing-recognizable 

Bogus “Proof”: Suppose for contradiction that there exists a recognizer \( R \) for \( A_{TM} \). We construct a recognizer for \( \overline{A_{TM}} \): 

On input \( \langle M, w \rangle \):
1. Run \( R \) on input \( \langle M, w \rangle \)
2. If \( R \) accepts, reject. Otherwise, accept.

This sure looks like a reduction from \( \overline{A_{TM}} \) to \( A_{TM} \)
Mapping Reductions: Motivation

1. How do we formalize the notion of a reduction?
2. How do we use reductions to show that languages are unrecognizable?
3. How do we protect ourselves from accidentally “proving” bogus statements about recognizability?
Computable Functions

Definition:

A function $f: \Sigma^* \to \Sigma^*$ is computable if there is a TM $M$ which, given as input any $w \in \Sigma^*$, halts with only $f(w)$ on its tape. ("Outputs $f(w)$")
Computable Functions

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\[
\langle x, y \rangle = x \# y
\]

Example 1: \( f(\langle x, y \rangle) = x + y \)

Example 2: \( f(\langle M, w \rangle) = \langle M' \rangle \) where \( M \) is a TM, \( w \) is a string, and \( M' \) is a TM that ignores its input and simulates running \( M \) on \( w \)

\[
M' = "On input x: \ \text{Ignore x, run M on w, output result}" \]
Mapping Reductions

Definition:
Language $A$ is mapping reducible to language $B$, written $A \leq_m B$ if there is a computable function $f: \Sigma^* \rightarrow \Sigma^*$ such that for all strings $w \in \Sigma^*$, we have $w \in A \iff f(w) \in B$. 
Mapping Reductions

Definition:

Language $A$ is mapping reducible to language $B$, written $A \leq_m B$, if there is a computable function $f : \Sigma^* \rightarrow \Sigma^*$ such that for all strings $w \in \Sigma^*$, we have $w \in A \iff f(w) \in B$.

Theorem: $A \leq_m B \implies \overline{A} \leq_m \overline{B}$.

If $A \leq_m B$, which of the following is always true?

a) $\overline{A} \leq_m B$

b) $A \leq_m \overline{B}$

c) $\overline{A} \leq_m \overline{B}$

d) $\overline{B} \leq_m \overline{A}$
Decidability

Theorem: If $A \leq_m B$ and $B$ is decidable, then $A$ is also decidable

Proof: Let $M$ be a decider for $B$ and let $f : \Sigma^* \rightarrow \Sigma^*$ be a mapping reduction from $A$ to $B$. Construct a decider for $A$ as follows:

On input $w$:
1. Compute $f(w)$
2. Run $M$ on input $f(w)$
3. If $M$ accepts, accept. If it rejects, reject.

Proof of correctness:

1) If $w \in A$, then $f(w) \in B$ [mapping rd.]
   $\Rightarrow M$ accepts on input $f(w)$ [M decides $B$]
   $\Rightarrow N$ accepts $w$

2) If $w \notin A$, then $f(w) \notin B$
   $\Rightarrow M$ rejects on input $f(w)$
   $\Rightarrow N$ rejects $w$
Undecidability

Theorem: If \( A \leq_m B \) and \( B \) is decidable, then \( A \) is also decidable

Corollary: If \( A \leq_m B \) and \( A \) is undecidable, then \( B \) is also undecidable
Old Proof: Equality Testing for TMs

\[ EQ_{TM} = \{ \langle M_1, M_2 \rangle | M_1, M_2 \text{ are TMs and } L(M_1) = L(M_2) \} \]

Theorem: \( EQ_{TM} \) is undecidable

Proof: Suppose for contradiction that there exists a decider \( R \) for \( EQ_{TM} \). We construct a decider for \( EQ_{TM} \) as follows:

On input \( \langle M \rangle \):
1. Construct TMs \( M_1, M_2 \) as follows:
   \[ M_1 = M \]
   \[ M_2 = \text{"On input } x, \text{ 1. Ignore } x \text{ and reject"} \]
2. Run \( R \) on input \( \langle M_1, M_2 \rangle \)
3. If \( R \) accepts, accept. Otherwise, reject.

This is a reduction from \( E_{TM} \) to \( EQ_{TM} \)
New Proof: Equality Testing for TMs

\[ EQ_{TM} = \{\langle M_1, M_2 \rangle \mid M_1, M_2 \text{ are TMs and } L(M_1) = L(M_2) \} \]

Theorem: \( E_{TM} \leq_m EQ_{TM} \) hence \( EQ_{TM} \) is undecidable

Proof: The following TM \( N \) computes the reduction \( f \):

\( E_{TM} \text{ undecidable to prove new statement that } EQ_{TM} \text{ undec} \)

On input \( \langle M \rangle \):

1. Construct TMs \( M_1, M_2 \) as follows:
   \[ M_1 = M \]
   \[ M_2 = "\text{On input } x, 1. \text{Ignore } x \text{ and reject}" \]

2. Output \( \langle M_1, M_2 \rangle \)

Function \( f \) : \( f(\langle M \rangle) = \langle M_1, M_2 \rangle \)

\[ L(M) = \emptyset \iff L(M_1) = L(M_2) \]

\[ \langle M \rangle \in E_{TM} \iff \langle M_1, M_2 \rangle \in EQ_{TM} \]
Mapping Reductions: Recognizability

Theorem: If \( A \leq_m B \) and \( B \) is recognizable, then \( A \) is also recognizable

Proof: Let \( M \) be a recognizer for \( B \) and let \( f : \Sigma^* \rightarrow \Sigma^* \) be a mapping reduction from \( A \) to \( B \). Construct a recognizer for \( A \) as follows:

1. Compute \( f(w) \)
2. Run \( M \) on input \( f(w) \)
3. If \( M \) accepts, accept. Otherwise, reject.
Unrecognizability

Theorem: If $A \leq_m B$ and $B$ is recognizable, then $A$ is also recognizable.

Corollary: If $A \leq_m B$ and $A$ is unrecognizable, then $B$ is also unrecognizable.

Corollary: If $A_{\text{TM}} \leq_m B$, then $B$ is unrecognizable.

Corollary: If $A_{\text{TM}} \leq_m \overline{B}$ then $B$ is unrecognizable.
Recognizability and $A_{TM}$

Let $L$ be a language. Which of the following is true?

a) If $L \leq_m A_{TM}$, then $L$ is recognizable
b) If $A_{TM} \leq_m L$, then $L$ is recognizable

Theorem: $L$ is recognizable if and only if $L \leq_m A_{TM}$
Recognizability and $A_{TM}$

Theorem: $L$ is recognizable if and only if $L \leq_m A_{TM}$

Proof:

$\leftarrow$ Follow from $A_{TM}$ recognizable

$\rightarrow$ Let $L$ be recognizable by TM $M$. Goal: Construct mapping reduction $f$ from $L$ to $A_{TM}$. The following TM $N$ computes $f$:

On input $w$ (instance of $L$)

Output $\langle M, w \rangle$

Analysis: wts that $w \in L \iff f(w) \in A_{TM}$

1) $w \in L \Rightarrow M$ accepts on input $w$ \[ \iff f(w) = \langle M, w \rangle \in A_{TM} \]

2) $w \notin L \Rightarrow M$ does not accept $w$

$\Rightarrow f(w) = \langle M, w \rangle \notin A_{TM}$