BU CS 332 – Theory of Computation

Lecture 18:

- More Mapping Reductions
- Computation History Method

Reading:

Sipser Ch 5.3, 5.1

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Mapping Reductions

Definition:

Language A is mapping reducible to language B, written

$$A \leq_{\mathrm{m}} B$$

if there is a computable function $f: \Sigma^* \to \Sigma^*$ such that for all strings $w \in \Sigma^*$, we have $w \in A \Leftrightarrow f(w) \in B$

Mapping Reductions: Implications

Theorem:

If $A \leq_{\mathrm{m}} B$ and B is decidable (resp. recognizable), then A is also decidable (resp. recognizable)

Corollary:

If $A \leq_{\mathrm{m}} B$ and A is undecidable (resp. unrecognizable), then B is also undecidable (resp. unrecognizable)

Example: Another reduction to EQ_{TM}

$$EQ_{\mathrm{TM}} = \{\langle M_1, M_2\rangle \mid M_1, M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$$

Theorem: $A_{\mathrm{TM}} \leq_{\mathrm{m}} EQ_{\mathrm{TM}}$

Proof: The following TM N computes the reduction f:

What should the inputs and outputs to f be?

a) f should take as input a pair $\langle M_1, M_2\rangle$ and output a pair $\langle M, w\rangle$
b) f should take as input a pair $\langle M, w\rangle$ and output a pair $\langle M_1, M_2\rangle$
c) f should take as input a pair $\langle M_1, M_2\rangle$ and either accept or reject
d) f should take as input a pair $\langle M, w\rangle$ and either accept or reject

Example: Another reduction to EQ_{TM}

$EQ_{\text{TM}} = \{\langle M_1, M_2\rangle \mid M_1, M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

Theorem: $A_{\text{TM}} \leq_{\text{m}} EQ_{\text{TM}}$

Proof: The following TM N computes the reduction f:

On input $\langle M, w\rangle$:

1. Construct TMs M_1, M_2 as follows:

M_1 = "On input x,

M_2 = "On input x,

2. Output $\langle M_1, M_2\rangle$

Consequences of $A_{\mathrm{TM}} \leq_{\mathrm{m}} EQ_{\mathrm{TM}}$

1. Since A_{TM} is undecidable, EQ_{TM} is also undecidable

2. $A_{\mathrm{TM}} \leq_{\mathrm{m}} EQ_{\mathrm{TM}}$ implies $\overline{A_{\mathrm{TM}}} \leq_{\mathrm{m}} \overline{EQ_{\mathrm{TM}}}$
   Since $\overline{A_{\mathrm{TM}}}$ is unrecognizable, $\overline{EQ_{\mathrm{TM}}}$ is unrecognizable

EQ_{TM} itself is also unrecognizable

$$EQ_{\mathrm{TM}} = \{\langle M_1, M_2\rangle \mid M_1, M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$$

Theorem: $\overline{A_{\mathrm{TM}}} \leq_{\mathrm{m}} EQ_{\mathrm{TM}}$ hence EQ_{TM} is unrecognizable

Proof: The following TM computes the reduction:

On input $\langle M, w\rangle$:

1. Construct TMs M_1, M_2 as follows:

   M_1 = "On input x,
   1. Ignore x
   2. Run M on input w
   3. If M accepts, accept.
      Otherwise, reject."

   M_2 = "On input x,
   1. Ignore x and reject"

2. Output $\langle M_1, M_2\rangle$

Computation History Method

Problems in Language Theory

Apparent dichotomy:

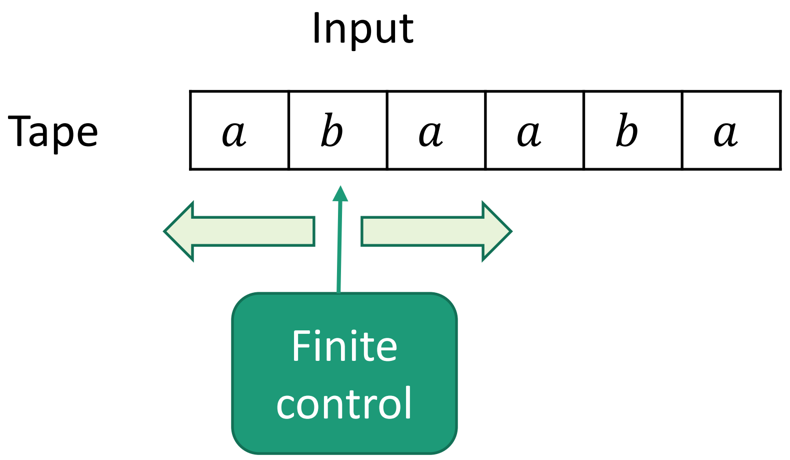
- TMs seem to be able to solve problems about the power of weaker computational models (e.g., DFAs)
- TMs can't solve problems about the power of TMs themselves

Question: Are there undecidable problems that do not involve TM descriptions?

A_{DFA} decidable	A_{TM} undecidable
E_{DFA} decidable	E_{TM} undecidable
EQ_{DFA} decidable	EQ_{TM} undecidable

Linear Bounded Automata (LBA)

A linear bounded automaton (LBA) is a TM variant with a bounded tape. The number of tape cells is the length of the input.



Intermediate in power between DFAs and TMs:

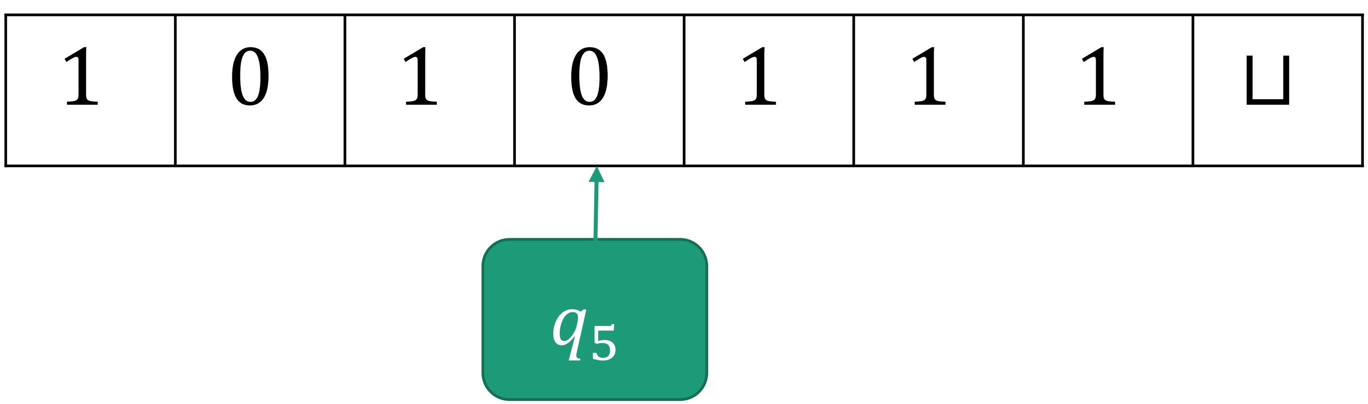
Regular langs. $\subsetneq$ $\mathrm{SPACE}(n)$ $\subsetneq$ Turing-recognizable langs.

Configurations

A configuration is a string uqv where $q \in Q$ and $u, v \in \Gamma^*$

- Tape contents = uv
- Current state = q
- Tape head on first symbol of v

Ex. $101q_5 0111 \sqcup$



Computing with Configurations

A sequence of configurations $C_0, \ldots, C_\ell$ is an accepting computation history for TM (or LBA) M on input w if

1. C_0 is the start configuration $q_0 w_1 \ldots w_n$
2. Every C_{i+1} legally follows from C_i
3. C_ℓ is an accepting configuration

Rejecting computation history: Same thing, but C_ℓ is a rejecting configuration

If M loops on w, there is no accepting or rejecting computation history

Counting Configurations

How many distinct configurations are possible for an LBA with k states, a symbols in its tape alphabet, and a tape of length n?

a. kan

b. $k + a + n$

c. ka^n

d. kna^n

LBA Halting

Theorem: Let B be an LBA with k states and a symbols in its tape alphabet. Then B halts on input w if and only if B halts on input w within kna^n steps.

Proof:

Deciding A_{LBA}

$A_{\text{LBA}} = \{\langle B, w\rangle \mid B \text{ is an LBA that accepts input } w\}$

Theorem: A_{LBA} is decidable

Proof: The following TM decides A_{LBA}:

On input $\langle B, w\rangle$:

1. Simulate B on input w **for $\boldsymbol{kna^n}$ steps**
2. If simulation accepts, accept.

   If simulation rejects or has not yet halted, reject.

LBAs can "check" TMs

LBAs are not powerful enough to perform general TM computations themselves.

But they can *check* the computation of a general TM M on input w

$B =$ "On input $x = \langle C_0, C_1, \ldots, C_\ell \rangle$ a sequence of configs.:

Accept if all of the following hold, and reject otherwise:

1. C_0 is the starting configuration of M on w,
2. Every C_{i+1} legally follows from C_i, and
3. C_ℓ is an accepting configuration"

What is the language of B?

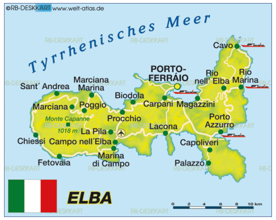
Computation History Method

Reduction from the undecidable language A_{TM} to a language L using the following idea:

Given an input $\langle M, w \rangle$ to A_{TM}, the ability to solve L enables checking the existence of an accepting computation history for M on w

Can be used to prove undecidability of E_{LBA}, ALL_{CFG}, Post Correspondence Problem, first-order logic …

E_{LBA} is unrecognizable



$E_{\mathrm{LBA}} = \{\langle B \rangle \mid B \text{ is an LBA recognizing } \emptyset\}$

Theorem: $\overline{A_{\mathrm{TM}}} \leq_{\mathrm{m}} E_{\mathrm{LBA}}$ hence E_{LBA} is unrecognizable

Proof: The following TM computes the reduction:

On input $\langle M, w \rangle$:

1. Construct LBA B as follows:

   $B =$ "On input $x = \langle C_0, C_1, \ldots, C_\ell \rangle$ a sequence of configs.:

   Accept if x is an accepting computation history of M on w. Otherwise, reject.

2. Output $\langle B \rangle$.

Recap of LBAs

LBAs are simple:

- Can determine whether an LBA halts on a given input by checking if it repeats a configuration
- Implies A_{LBA} is decidable

LBAs are powerful:

- An LBA can check the computation of a general TM on a given input
- Implies E_{LBA} is undecidable

Problems in Language Theory

A_{DFA} decidable	A_{LBA} decidable	A_{TM} undecidable
E_{DFA} decidable	E_{LBA} undecidable	E_{TM} undecidable
EQ_{DFA} decidable	EQ_{LBA} undecidable	EQ_{TM} undecidable

Undecidable problems outside language theory

Post Correspondence Problem (PCP):

Domino: $\left[\frac{a}{ab}\right]$. Top and bottom are strings.

Input: Collection of dominos.

$$\left[\frac{aa}{aba}\right], \left[\frac{ab}{aba}\right], \left[\frac{ba}{aa}\right], \left[\frac{abab}{b}\right]$$

Match: List of some of the input dominos (repetitions allowed) where top = bottom

$$\left[\frac{ab}{aba}\right], \left[\frac{aa}{aba}\right], \left[\frac{ba}{aa}\right], \left[\frac{aa}{aba}\right], \left[\frac{abab}{b}\right]$$

Problem: Does a match exist? This is undecidable