BU CS 332 – Theory of Computation

Lecture 21:

- Complexity Class P
- Nondeterministic time, NP

Reading:

Sipser Ch 7.2, 7.3

Mark Bun

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Complexity class P

Definition: P is the class of languages decidable in polynomial time on a basic single-tape (deterministic) TM

$$\mathrm{P} = \bigcup_{k=1}^{\infty} \mathrm{TIME}(n^k) = \mathrm{TIME}(n) \cup \mathrm{TIME}(n^2) \cup \mathrm{TIME}(n^3) \cup \ldots$$

- Class doesn't change if we substitute in another reasonable deterministic model (Extended Church-Turing)
- Cobham-Edmonds Thesis: Roughly captures class of problems that are feasible to solve on computers

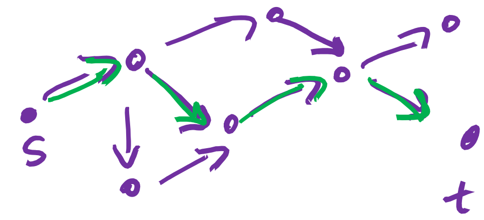
Describing and analyzing polynomial-time algorithms

- Due to Extended Church-Turing Thesis, we can still use high-level descriptions on multi-tape machines
- Polynomial-time is robust under composition: $\mathrm{poly}(n)$ executions of $\mathrm{poly}(n)$-time subroutines run on $\mathrm{poly}(n)$-size inputs gives an algorithm running in $\mathrm{poly}(n)$ time.

  $\Rightarrow$ Can freely use algorithms we've seen before as subroutines if we've analyzed their runtime

- Need to be careful about size of inputs! (Assume inputs represented in <u>binary</u> unless otherwise stated.)

Examples of languages in $\mathbf{P}$



$PATH =$
$\{\langle G, s, t\rangle \mid G \text{ is a directed graph with a directed path from } s \text{ to } t\}$

$\in P$

Poly-time alg. via BFS

On input $\langle G, s, t\rangle$:

1. Mark vertex s
2. Until no new vertices are being marked:
   - Mark all out-neighbors of currently marked vertices
3. Accept if t is marked and reject otherwise

Runs in time $O(|E|)$ (# edges)

Polynomial in $|\langle G, s, t\rangle|$

Examples of languages in $\mathbf{P}$

$E_{\mathrm{DFA}} = \{\langle D\rangle \mid D \text{ is a DFA that recognizes the empty language}\}$
$\in P$

- Also BFS

- Check at the end of BFS if any accept state has been marked

Examples of languages in $\mathbf{P}$

encoded in binary

- $RELPRIME = \{\langle x, y\rangle \mid x \text{ and } y \text{ are relatively prime}\}$
  - x, y relatively prime if $gcd(x,y) = 1$
  - Euclid's algorithm
- $PRIMES = \{\langle x\rangle \mid x \text{ is prime}\}$

2006 Gödel Prize citation



The 2006 Gödel Prize for outstanding articles in theoretical computer science is awarded to Manindra Agrawal, Neeraj Kayal, and Nitin Saxena for their paper "PRIMES is in P."

In August 2002 one of the most ancient computational problems was finally solved....

A polynomial-time algorithm for $PRIMES$?

Consider the following algorithm for $PRIMES$

On input $\langle x \rangle$: Encoded in binary

For $b = 2, 3, 4, 5, \ldots, \sqrt{x}$:

- Try to divide x by b
- If b divides x, accept

If all b fail to divide x, reject

Input length n

x could be as large as 2^n

divisions : $\sqrt{2^n} = 2^{n/2}$

How many divisions does this algorithm require in terms of $n = |\langle x \rangle|$? a) $O(\sqrt{n})$ b) $O(n)$ c) $2^{O(\sqrt{n})}$ d) $2^{O(n)}$

Beyond polynomial time

Definition: EXP is the class of languages decidable in exponential time on a basic single-tape (deterministic) TM

$$\mathrm{EXP} = \bigcup_{k=1}^{\infty} \mathrm{TIME}(2^{n^k})$$

$= \mathrm{TIME}(2^{n}) \cup \mathrm{TIME}(2^{n^2}) \cup \mathrm{TIME}(2^{n^3}) \cup \ldots$

Why study P ?

Criticism of the Cobham-Edmonds Thesis:

- Algorithms running in time n^{100} aren't really efficient

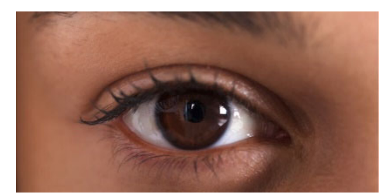
  Response: Runtimes improve with more research

- Does not capture some physically realizable models using randomness, quantum mechanics "hardness vs. randomness"

  Response: Randomness may not change P, useful principles

$TIME(n)$ vs. $TIME(n^2)$

P vs. EXP



decidable vs. undecidable

Nondeterministic Time and NP

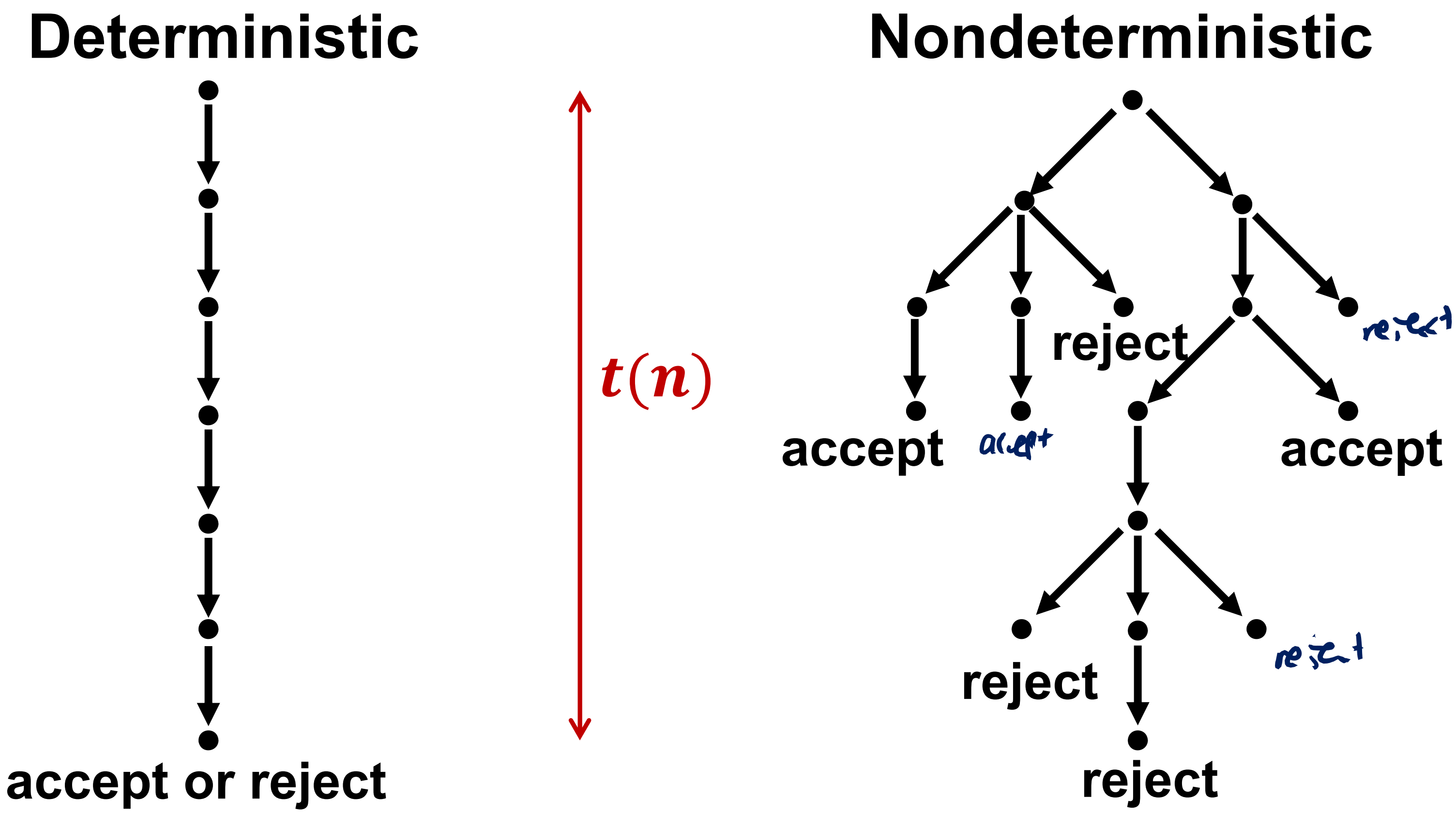
Nondeterministic time

input lengths　　worst-case runtimes

Let $f : \mathbb{N} \rightarrow \mathbb{N}$

A NTM M runs in time $f(n)$ if on every input $w \in \Sigma^n$, M halts on w within at most $f(n)$ steps on every computational branch

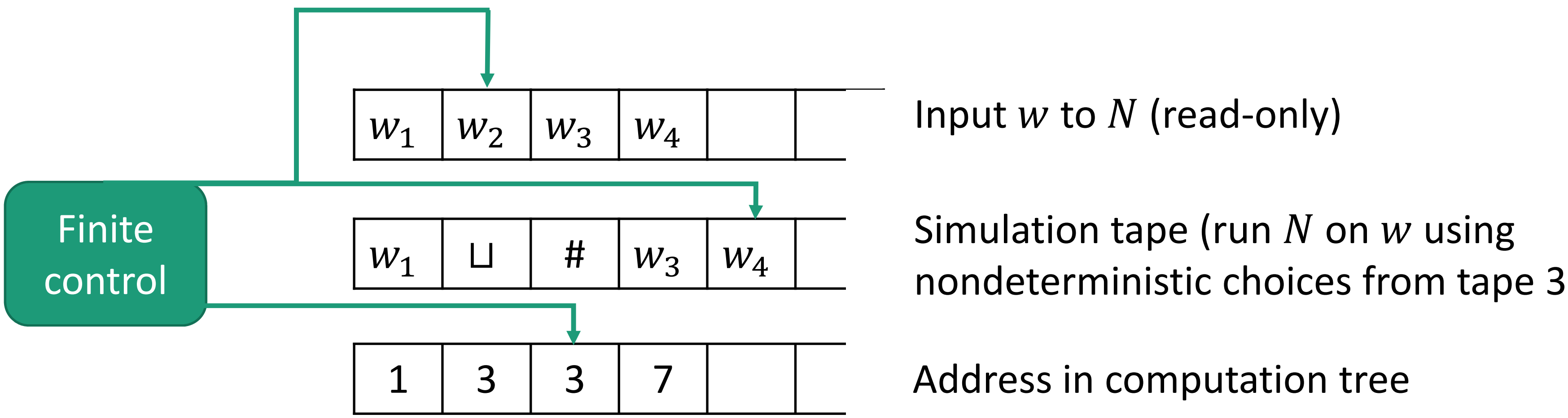
Deterministic vs. nondeterministic time



Deterministic vs. nondeterministic time

Theorem: Let $t(n) \geq n$ be a function. Every NTM running in time $t(n)$ has an equivalent single-tape TM running in time $2^{O(t(n))}$

Proof: Simulate NTM by 3-tape TM



Counting leaves

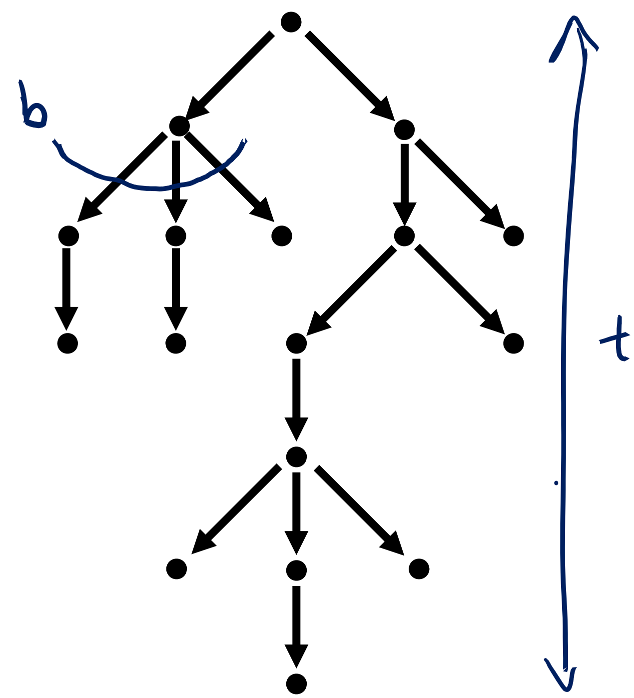
What is the maximum number of leaves in a tree with branching factor b and depth t?

a) bt

b) b^t

c) t^b

d) 2^t



Deterministic vs. nondeterministic time

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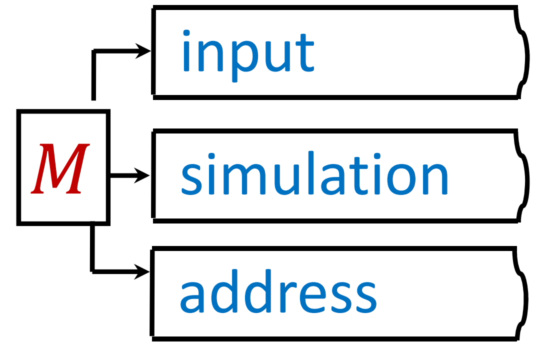
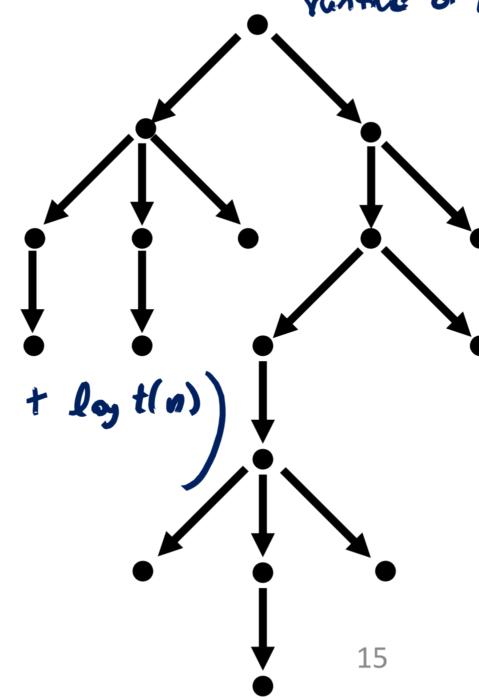


b = max # of nondet. choices in any step

t(n) = runtime of NTM

Proof: Simulate NTM by 3-tape TM

- # leaves: $b^{t(n)}$



Running time:

To simulate one root-to-leaf path:

$O(t(n))$

Total time: $O\left(t(n) \cdot b^{t(n)}\right) = O\left(2^{t(n)\log b + \log t(n)}\right)$

$= 2^{O(t(n))}$

Deterministic vs. nondeterministic time

Theorem: Let $t(n) \geq n$ be a function. Every NTM running in time $t(n)$ has an equivalent single-tape TM running in time $2^{O(t(n))}$

Proof: Simulate NTM by 3-tape TM in time $2^{O(t(n))}$

We know that a 3-tape TM can be simulated by a single-tape TM with quadratic overhead, hence we get running time

$$\left(2^{O(t(n))}\right)^2 = 2^{2 \cdot O(t(n))} = 2^{O(t(n))}$$

basic single-tape TM

Difference in time complexity

Extended Church-Turing Thesis:

At most polynomial difference in running time between all (reasonable) deterministic models

At most exponential difference in running time between deterministic and nondeterministic models

Nondeterministic time

Let $f : \mathbb{N} \rightarrow \mathbb{N}$

A NTM M runs in time $f(n)$ if on every input $w \in \Sigma^n$, M halts on w within at most $f(n)$ steps on every computational branch

$\mathrm{NTIME}(f(n))$ is a class (i.e., set) of languages:

A language $A \in \mathrm{NTIME}(f(n))$ if there exists an NTM M that

1) Decides A, and
2) Runs in time $O(f(n))$

NTIME explicitly

A language $A \in \text{NTIME}(f(n))$ if there exists an NTM M such that, on every input $w \in \Sigma^*$

1. Every computational branch of M halts in either the accept or reject state within $f(|w|)$ steps

2. $w \in A$ iff there exists an accepting computational branch of M on input w

3. $w \notin A$ iff every computational branch of M rejects on input w (or dies with no applicable transitions)

Complexity class NP

Definition: NP is the class of languages decidable in polynomial time on a nondeterministic TM

$$\text{NP} = \bigcup_{k=1}^{\infty} \text{NTIME}(n^k)$$

$= \text{NTIME}(n) \cup \text{NTIME}(n^2) \ldots$

Which of the following are definitely true about NP?

a) $\text{P} \subseteq \text{NP}$ ✓ det. alg's are non-det. algs

b) $\text{NP} \subseteq \text{P}$?

c) $\text{NP} \not\subseteq \text{P}$

win $1M if you answer this

d) $\text{NP} \subseteq \text{EXP}$ ✓ $\text{NTIME}(f(n)) \subseteq \text{TIME}(2^{O(f(n))})$

$\text{NTIME}(n^k) \subseteq \text{TIME}(2^{O(n^k)})$

e) $\text{EXP} \subseteq \text{NP}$?

$\text{TIME}(f(n)) =$ languages decidable in $f(n)$ steps on a DTM

$\text{NTIME}(f(n)) =$ languages " " on an NTM

Hamiltonian Path

$HAMPATH = \{\langle G, s, t\rangle \mid G$ is a directed graph and there is a path from s to t that passes through every vertex exactly once$\}$

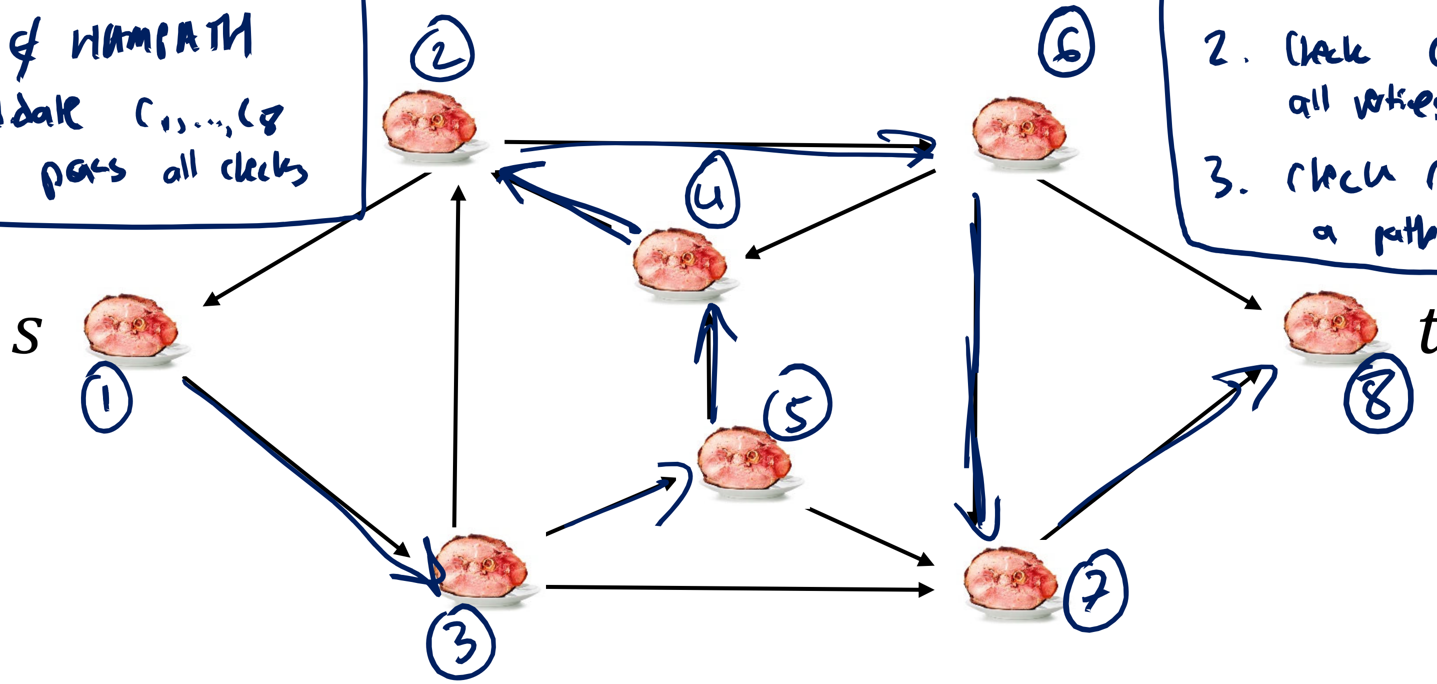
$\langle G,s,t\rangle \in HAMPATH$:

$c_1 = 1,\ c_2 = 3,\ c_3 = 5, \ldots$ passes all checks

$\langle G,s,t\rangle \notin HAMPATH$

No candidate $c_1, \ldots, c_8$ could pass all checks

1. Guess $c_1, \ldots, c_8$
2. Check $c_1, \ldots, c_8$ hits all vertices $1, \ldots, 8$
3. Check $c_1, \ldots, c_8$ is a path from s to t



$HAMPATH \in \text{NP}$

The following nondeterministic algorithm decides $HAMPATH$ in polynomial time:

• Adjacency matrix $|V|^2$, adjacency list $|E| + |V|$

On input $\langle G, s, t \rangle$: (Vertices of G are numbers $1, \ldots, k$)

1. **Nondeterministically** guess a sequence $c_1, c_2, \ldots, c_k$ of numbers $1, \ldots, k$
2. Check that $c_1, c_2, \ldots, c_k$ is a permutation: Every number $1, \ldots, k$ appears exactly once
3. Check that $c_1 = s$, $c_k = t$, and there is an edge from every c_i to c_{i+1}
4. Accept if all checks pass, otherwise, reject.

Analyzing the algorithm

Need to check:

1) Correctness

- $\langle G,s,t\rangle \in HAMPATH \Rightarrow \exists$ an accepting comp. branch
- $\langle G,s,t\rangle \notin HAMPATH \Rightarrow \forall$ comp. branches reject

2) Running time

- Show poly time on NTM
- (Check all comp. branches halt in poly time)