BU CS 332 – Theory of Computation

Lecture 25:

- Final review

Reading:

Sipser Ch 7.1-8.2, 9.1

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April 28, 2021

- HW 9 due Friday 11:59pm
- Test out tomorrow, due Thu 5/6, 5PM
- Normal OH during test week (except Friday)

Final Topics

Everything from Midterms 1 and 2

- Midterm 1 topics: DFAs, NFAs, regular expressions, distinguishing set method
  (more detail in lecture 8 notes)

- Midterm 2 topics: Turing machines, TM variants, Church-Turing thesis, decidable languages, countable and uncountable sets, undecidability, reductions, unrecognizability
  (more detail in lecture 16 notes)

Mapping Reducibility (5.3)

- Understand the definition of a computable function
- Understand the definition of a mapping reduction
- Know how to use mapping reductions to prove decidability, undecidability, recognizability, and unrecognizability

$A \leq_m B$

B decidable => A decidable

A undecidable => B undecidable

Time and Space Complexity (7.1)

+ 8.1

- Asymptotic notation: Big-Oh, little-oh
- Know the definition of running time and space for a TM and of time and space complexity classes (TIME / NTIME / SPACE / NSPACE)
- Understand how to simulate multi-tape TMs and NTMs using single-tape TMs and know how to analyze the running time overhead

P and NP (7.2, 7.3)

- Know the definitions of P and NP as time complexity classes
- Know how to analyze the running time of algorithms to show that languages are in P / NP
- Understand the verifier interpretation of NP and why it is equivalent to the NTM definition
- Know how to construct verifiers and analyze their runtime

NP-Completeness (7.4, 7.5)

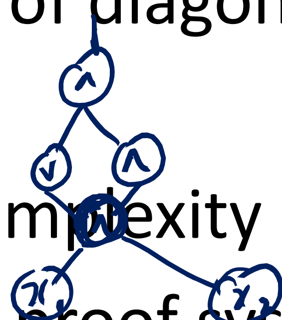
- Know the definition of poly-time reducibility
- Understand the definitions of NP-hardness and NP-completeness
- Understand the statement of the Cook-Levin theorem (don't need to know its proof)
- Understand several canonical NP-complete problems and the relevant reductions: SAT, 3SAT, CLIQUE, INDEPENDENT-SET, VERTEX-COVER, HAMPATH, SUBSET-SUM

Hierarchy Theorems (9.1)

- Formal statements of time and space hierarchy theorems and how to apply them
- How to use hierarchy theorems to prove statements like $\mathrm{P} \neq \mathrm{EXP}$

Things we didn't get to talk about

- Additional classes between NP and PSPACE (polynomial hierarchy)
- Logarithmic space
- Relativization and the limits of diagonalization
- Boolean circuits
- Randomized algorithms / complexity classes
- Interactive and probabilistic proof systems
- Complexity of counting



https://cs-people.bu.edu/mbun/courses/535_F20/

Theory and Algorithms Courses after 332

- Algorithms
  - CS 530/630 (Advanced algorithms)
  - CS 531 (Optimization algorithms)
  - CS 537 (Randomized algorithms)
- Complexity
  - CS 535 (Complexity theory)
- Cryptography
  - CS 538 (Foundations of crypto)
- Topics (CS 591)

  E.g., Privacy in machine learning, algorithms and society, sublinear algorithms, new developments in theory of computing, communication complexity

Algorithms and Theory Research Group

- https://www.bu.edu/cs/research/theory/

- Weekly seminar: Mondays at 11
  https://www.bu.edu/cs/algorithms-and-theory-seminar/

  Great way to learn about research in theory of computation!

Tips for Preparing Exam Solutions

Designing (nondeterministic) time/space-bounded deciders

describe algorithm

The following algorithm decides EC in polynomial time:

"On input $\langle A, C, e, p \rangle$, four binary integers:

1. Let $r \leftarrow 1$.

...

6. If $C = r \mod p$, *accept*; otherwise *reject*."

The algorithm is called *repeated squaring*.

Analysis of runtime

Let $T(d)$ denote a polynomial upper bound on the running time of basic procedures for multiplication and modular operations on d-bit numbers. Then steps 4 and 5 of the algorithm each take at most $O(T(\log A) + T(\log p))$ time because r is never larger than p. In addition, the total number of multiplication and modular operations is $O(k) = O(\log e)$. Therefore, the total running time of the algorithm is polynomial in $O((\log e) \cdot (T(\log A) + T(\log p)))$ which is polynomial in n. Hence, the total running time is polynomial. Note that without performing $\mod p$ operation in Steps 4 and 5,

Correctness →

- Key components: High-level description of algorithm, explanation of correctness, analysis of running time and/or space usage

Designing NP verifiers

certificate

We give a poly-time verifier for $TEAM$. A certificate c for our verifier is a subset of M of size k.

describe alg

"On input $\langle n, X, Y, Z, M, k; c\rangle$ where $\langle n, X, Y, Z, M, k\rangle$ is a $TEAM$ instance and c is a certificate:

1. If $k > \min(n, |M|)$, *reject*.
2. Check whether $|c| = k$ and $c \subseteq M$.
3. Check whether all elements of triples in c are different.
4. If any of these checks fails, *reject*; otherwise, *accept*."

analyze runtime

Step 1 is performed to ensure that the running time is polynomial in n even for large k. Step 2 can be run in $O(k \cdot |M|) = O(|M|^2)$ time, by iterating through M and marking elements. Step 3 can be implemented to run in $O(|c| \log |c|)$ time by first sorting the elements of c. This verifier runs in polynomial time; hence, $TEAM \in$ NP.

explain correctness

- Key components: Description of certificate, high-level description of algorithm, explanation of correctness, analysis of running time

NP-completeness proofs

To show a language L is NP-complete:

1) Show L is in NP (follow guidelines from previous two slides)

2) Show L is NP-hard (usually) by giving a poly-time reduction $A \leq_p L$ for some NP-complete language A

Double check you're doing reduction in correct direction!

- High-level description of algorithm computing reduction
- Explanation of correctness: Why is $w \in A$ iff $f(w) \in L$ for your reduction f?
- Analysis of running time

Practice Problems

When the encoding matters:

Aside: TMSAT = $\{\langle N, w, 1^t\rangle \mid \ldots\}$

Primality testing: Given nat. number x, is x prime?

UNARYPRIMES = $\{1^n \mid n \text{ is prime}\}$

PRIMES = $\{\langle x\rangle \mid x \text{ is prime}\}$

($\langle x\rangle$: binary encoding)

Alg. for UNARYPRIMES:

On input 1^n:

For each $y = 2, 3, 4, \ldots, n-1$:

Reject if y divides n

Accept

Running time = $O(n) \cdot poly(n)$

$= poly(n)$

($O(n)$: # times through main loop; $poly(n)$: Arithmetic computation)

Alg. for PRIMES:

On input $\langle x\rangle$:

For each $y = \ldots$

Reject ...

Accept ...

Running time = $O(x) \cdot poly(|\langle x\rangle|)$

Encoding length of input

$n \leq \log x$

$\Rightarrow$ runtime $O(x) = O(2^n)$

DOUBLE-SAT = { ⟨φ⟩ | φ is a general boolean formula w/ ≥ 2 satisfying assignments }

i.e. $\exists x_1, \ldots, x_n \neq y_1, \ldots, y_n$ s.t. $\varphi(x) = \varphi(y) = 1$

Claim: DOUBLE-SAT is NP-complete

1) DOUBLE-SAT ∈ NP

Design NP verifier. Certificate: two sat. assm'ts $b_1, \ldots, b_n$, $c_1, \ldots, c_n$

Verifier: on input φ (n variables), $\vec{b}, \vec{c}$

1. Check $\vec{b} \neq \vec{c}$
2. Evaluate $\varphi(\vec{b})$, $\varphi(\vec{c})$. Accept iff both evaluate to true. (Reject o.w.)

Correctness: • ∃ two satisfying assm'ts, certificate consisting of these makes verifier accept,

• < 2 sat assm'ts ⇒ no way to get verifier to accept on cert. $\vec{b}, \vec{c}$

Runtime: • φ can be evaluated in poly time, certificate is just 2 assm't's

2) DOUBLE-SAT is NP-hard

via SAT $\leq_p$ DOUBLE-SAT

On input $\langle \varphi \rangle$ (φ an n-variable formula $\varphi(x_1, \ldots, x_n)$)

1. $\psi(x_1, \ldots, x_n, y) = \varphi(x_1, \ldots, x_n)$

Output $\langle \psi \rangle$ (ψ an m-variable formula)

Ex: $\varphi(x_1, x_2) = x_1 \wedge x_2$

$\psi(x_1, x_2, y) = x_1 \wedge x_2$

Claim: φ has ≥ 1 sat. assm't $\Leftrightarrow$ ψ has ≥ 2 sat. assm'ts

$\Rightarrow$] if b is a sat. asm't to φ, then ($\vec{x} = \vec{b}$, $y = 0$) and ($\vec{x} = \vec{b}$, $y = 1$) are both sat. assm'ts to ψ

$\Leftarrow$] if φ has no sat. assm'ts ($\varphi(x_1, \ldots, x_n)$ un sat.), so $\psi(x_1, \ldots, x_n, y)$ un sat.

Runtime: linear time (copy formula)

Ex:

$\varphi(x_1, x_2) = x_1 \wedge x_2$ $\vec{b} = (1, 1)$

$\psi(x_1, x_2, y) = x_1 \wedge x_2$

$x_1 \; x_2 \; y$

$(1, 1, 0)$

$(1, 1, 1)$

Claim: If $P \neq NP$ and $A \in P$, then A is not NP-complete

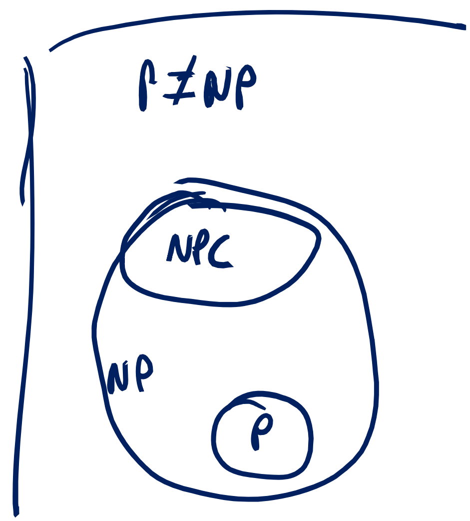
Proof. $\Leftarrow$ If $A \in P$ and A is NP-complete, then $P = NP$

- $A \in NP$ (duh)
- $\forall B \in NP$, $B \leq_p A$ (A is NP-hard)

Recall **Thm:** If $B \leq_p A$, $A \in P$, then $B \in P$

$\Rightarrow$ $\forall B \in NP$, $B \in P$ $\Rightarrow$ $NP \subseteq P$

$\therefore \Rightarrow$ $P = NP$



Use a mapping reduction to show that $ALL_{\mathrm{TM}} = \{\langle M \rangle | M \text{ is a TM and } L(M) = \Sigma^*\}$ is co-unrecognizable

Use a mapping reduction to show that $ALL_{\mathrm{TM}} = \{\langle M \rangle | M \text{ is a TM and } L(M) = \Sigma^*\}$ is unrecognizable

Give examples of the following languages: 1) A language in P. 2) A decidable language that is not in P. 3) A language for which it is unknown whether it is in P.

Give an example of a problem that is solvable in polynomial-time, but which is not in P

Let $L = \{\langle w_1, w_2\rangle | \exists \text{ strings } x, y, z \text{ such that } w_1 = xyz \text{ and } w_2 = xy^R z\}$. Show that $L \in \mathrm{P}$.

Which of the following operations is P closed under? Union, concatenation, star, intersection, complement.

Prove that $LPATH = \{\langle G, s, t, k\rangle | G \text{ is an undirected graph containing a simple path from } s \text{ to } t \text{ of length} \geq k\}$ is in NP

Prove that $LPATH$ is NP-hard

Which of the following operations is NP closed under? Union, concatenation, star, intersection, complement.

Which of the following statements are true?

- $SPACE(2^n) = SPACE(2^{n+1})$

- $SPACE(2^n) = SPACE(3^n)$

- $NSPACE(n^2) = SPACE(n^5)$